

Chaos
Sensitivity to initial data
Jim Yorke
University of Maryland

Movies on Sensitivity

- Sliding Doors 2002 Gwyneth Paltrow
- Run Lola Run 1999
- Ray Bradbury 1952 → the Simpsons
 - “A Sound of Thunder” short story
- Lorenz 1979: “Does the flap of a butterfly’s wings in Brazil set off a tornado in Texas.”

Sensitivity Proverb

For Want of a Nail

For want of a nail the shoe was lost.

For want of a shoe the horse was lost.

For want of a horse the rider was lost.

For want of a rider the battle was lost.

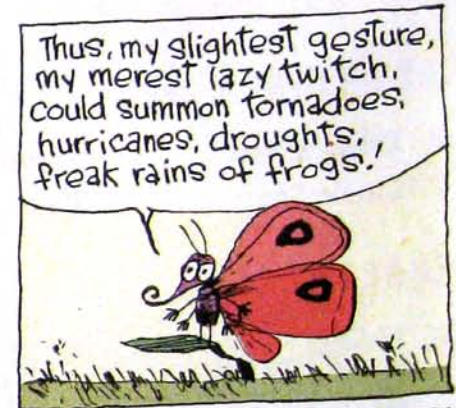
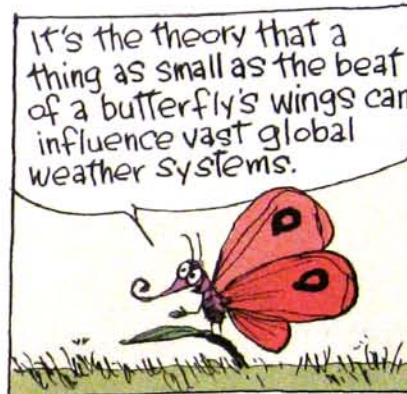
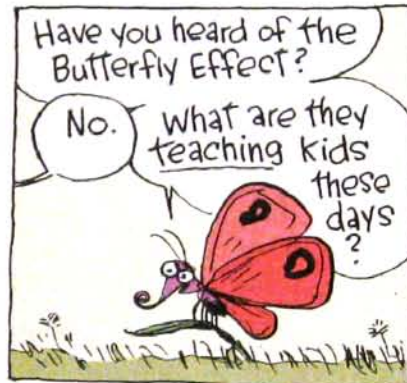
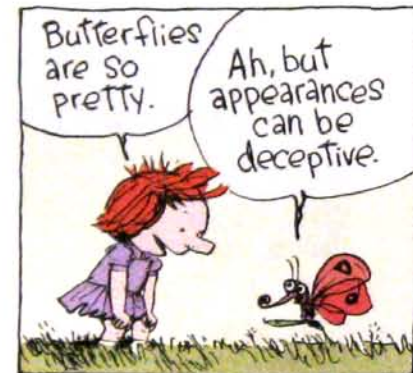
For want of a battle the kingdom was lost.

And all for the want of a horseshoe nail

Sensitivity to initial data

CUL DE SAC

by Richard Thompson



Cul de Sac is online at washingtonpost.com/magazine.

Three kinds of behavior for bounded dynamical behavior (ignoring transients)

- A. Steady state
- B. Periodic or quasiperiodic
- C. Chaotic (like double pendulum)

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$1.32 \pmod{1} = .32$

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Three kinds of behavior for bounded dynamical behavior

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.33	$1/3$	$1/300$
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.64	$2/3$	$8/300$
.28	$1/3$	$16/300$
.56	...	$32/300$
.12		$64/300$

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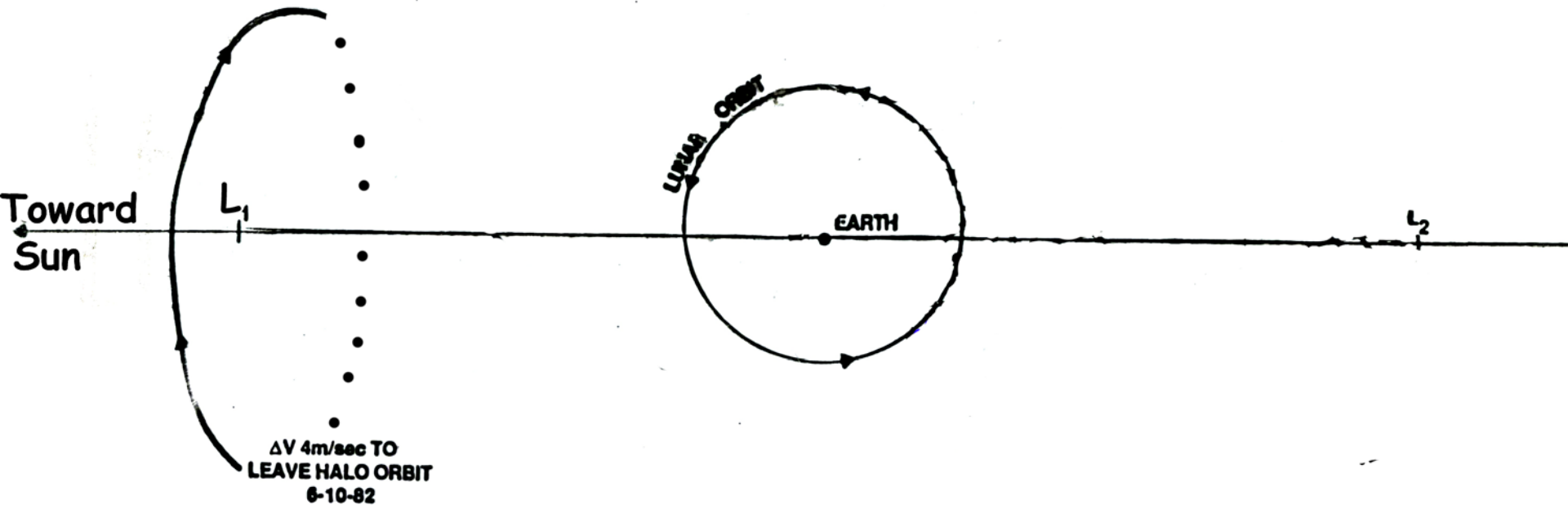
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ISEE Multiple Lunar Swingby Trajectory

First Spacecraft to fly through the tail of a comet

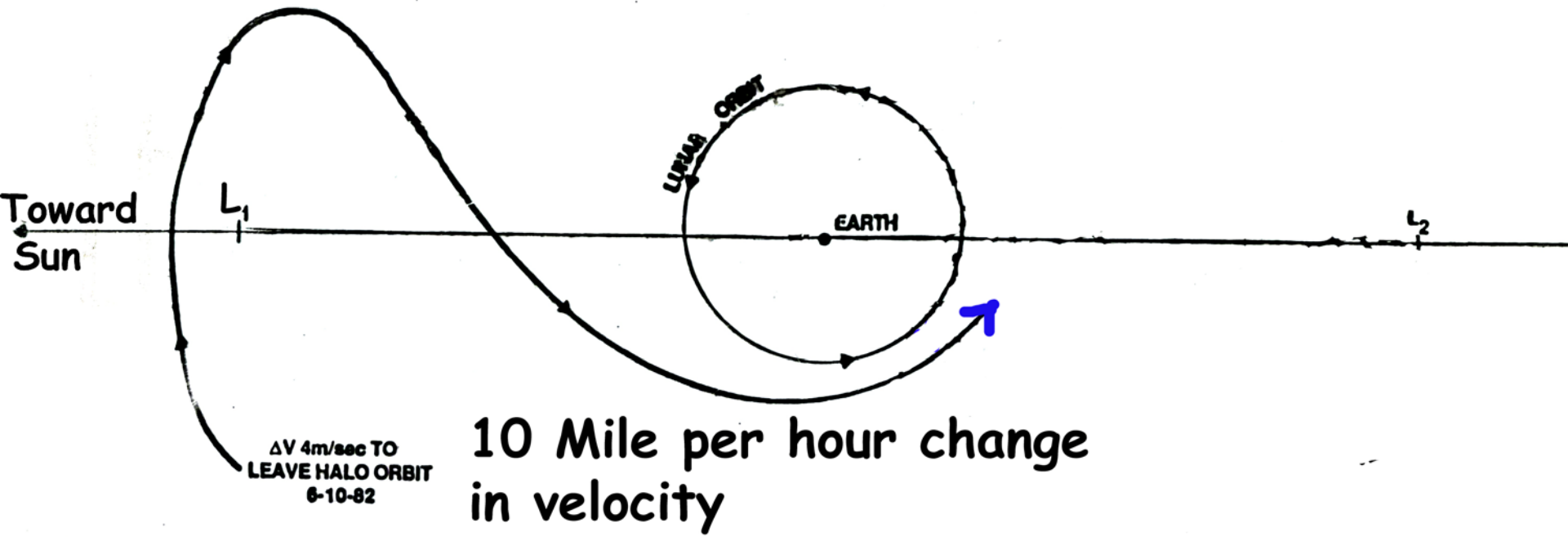
Rotating coordinates:

The sun always appears to be
in the same direction



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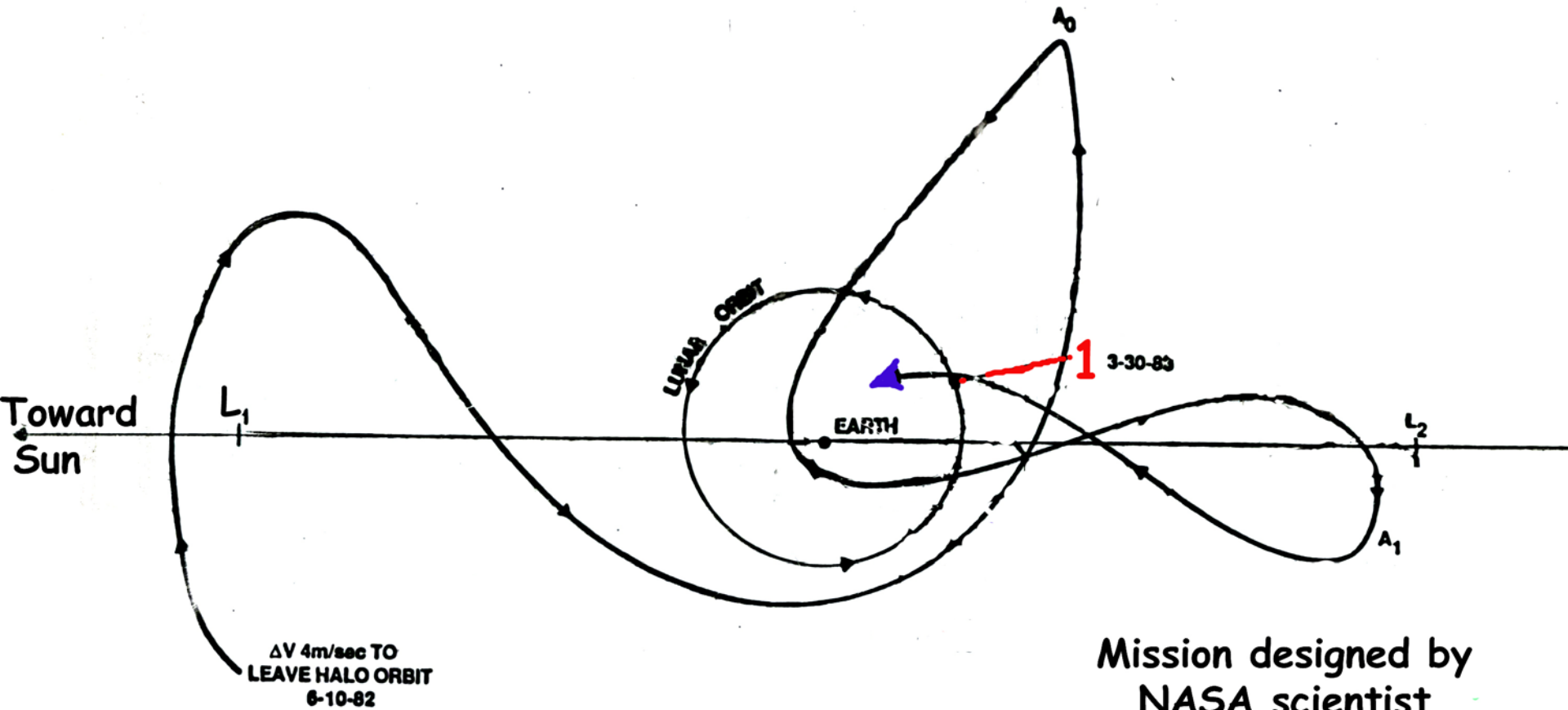


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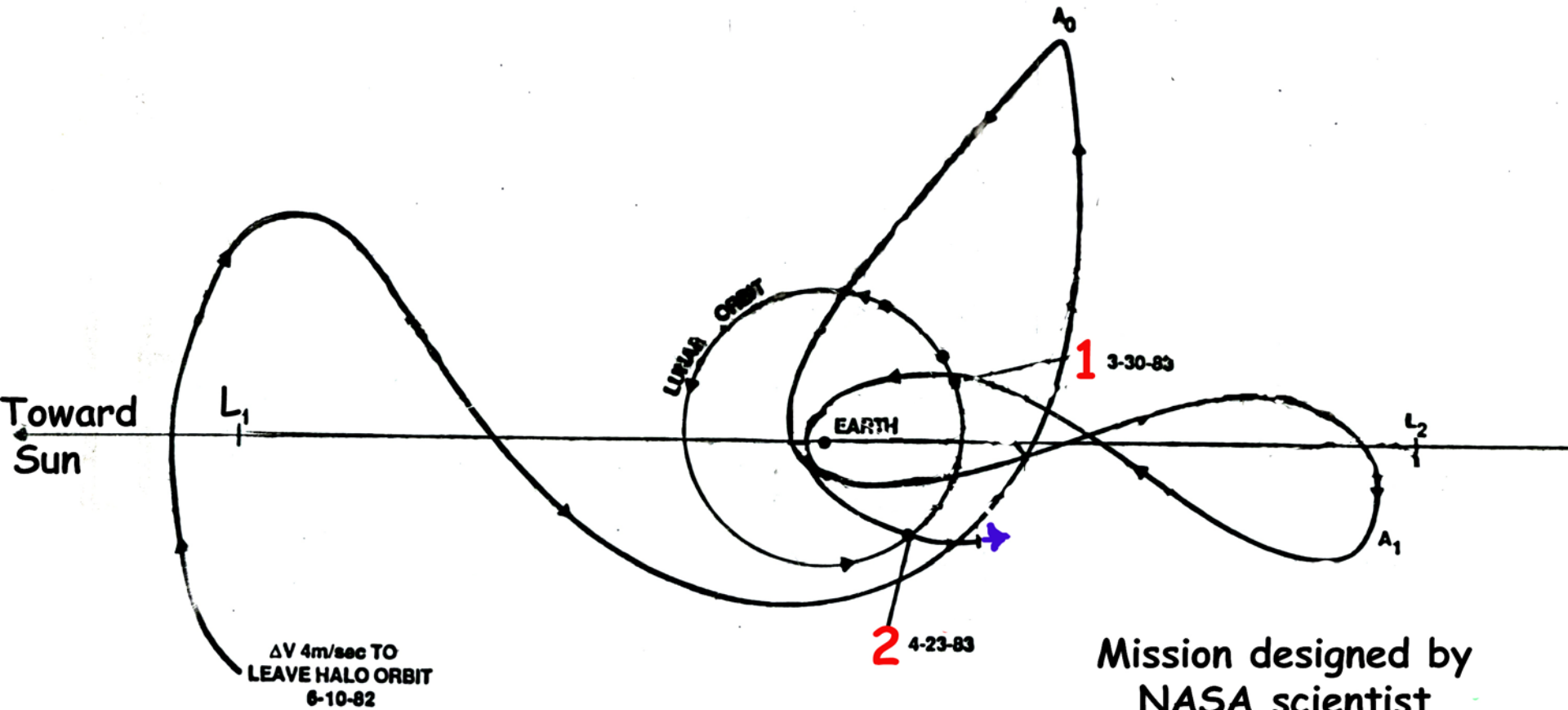
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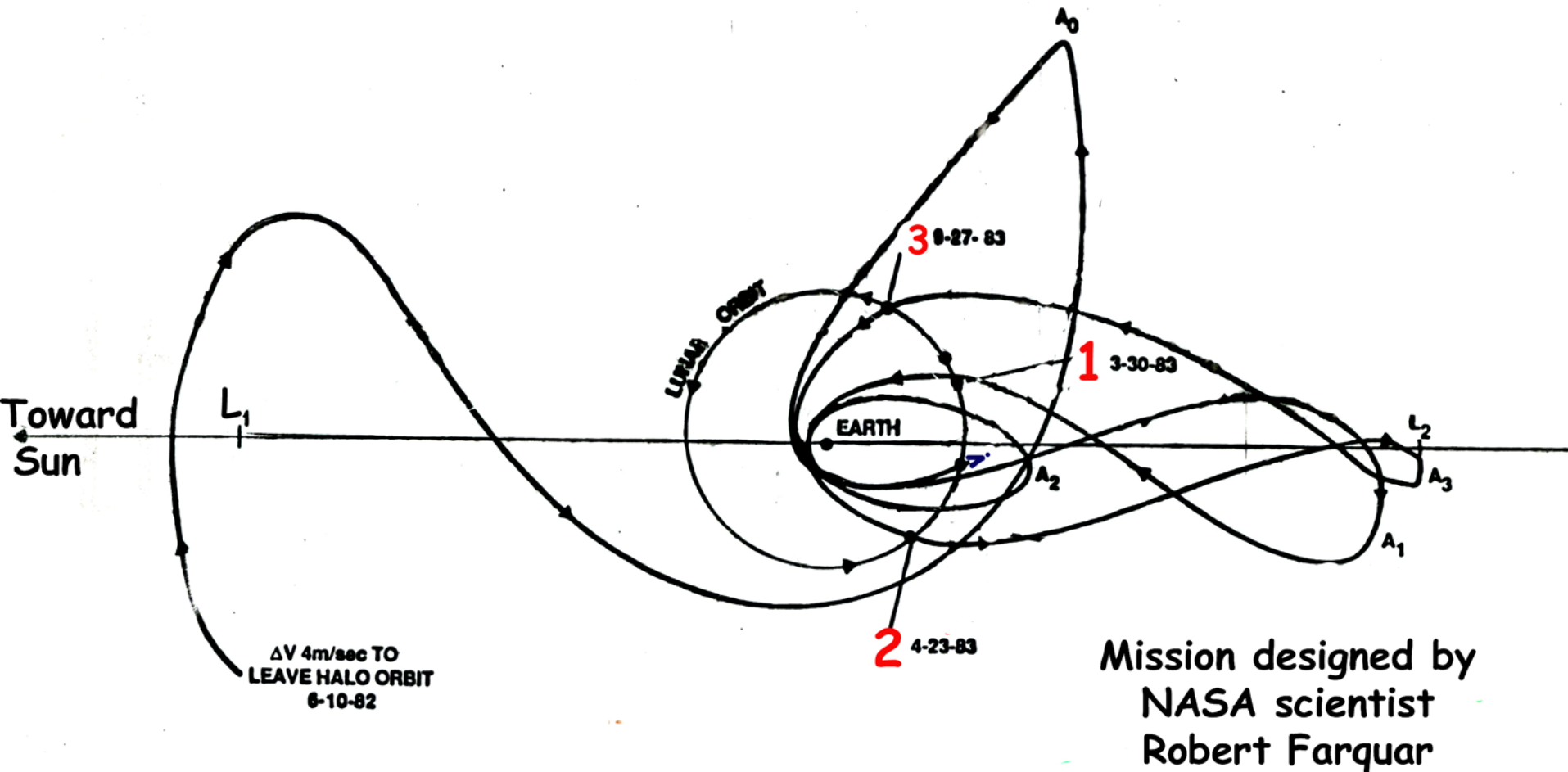
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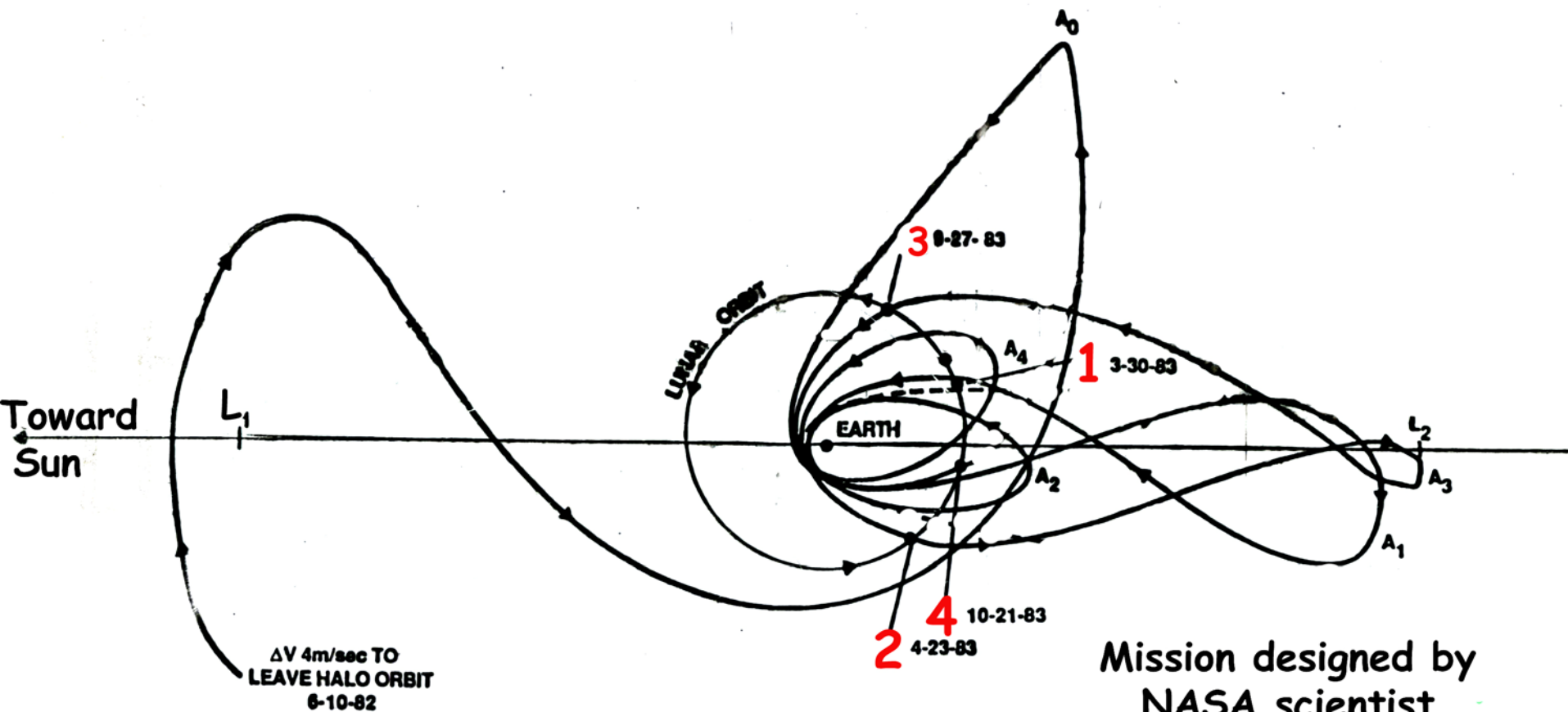
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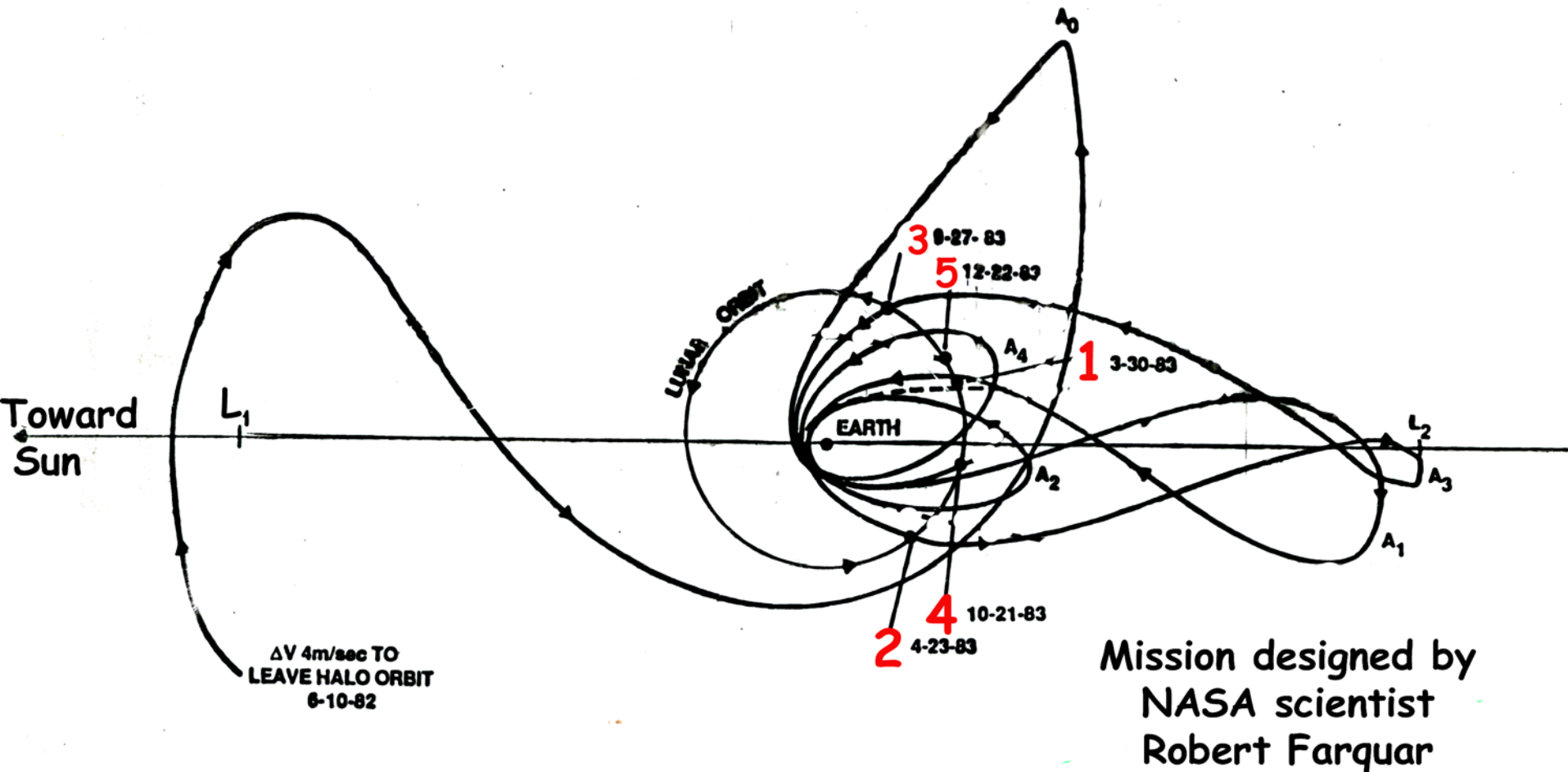
First Spacecraft to fly through the tail of a comet



Mission designed by
NASA scientist
Robert Farquar

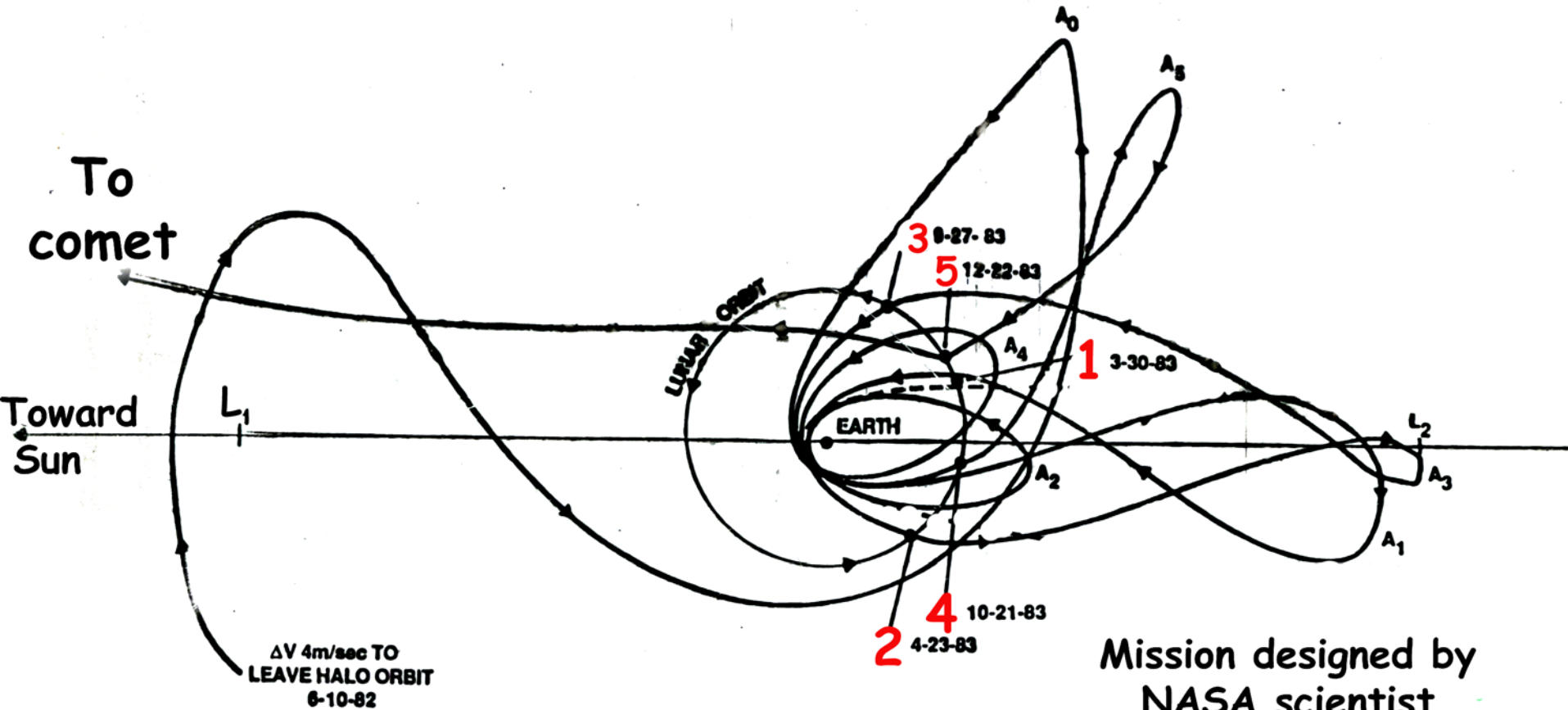
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Circular planar restricted 3-body problem

The **restricted three-body problem** assumes that the mass of one of the bodies is negligible;

the **circular** restricted three-body problem is the special case in which two of the bodies are in circular orbits

(approximated by the **spacecraft - Earth – Moon** system).

If the time for one revolution is 2π ,
then an invariant of the motion is the

[Kinetic + potential energy] – angular momentum

Hence the space craft can gain total energy if it also gains angular momentum (as measured from center of mass of the system)

Basins of attraction

- A basin is the set of initial points whose trajectory goes to a specified attractor.
- In practice we may choose a small disk around an attracting periodic point and compute which initial points (in a grid) whose trajectories enter that disk.

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Forced Damped Pendulum basins of attraction

- $\theta'' + a \theta' + \sin \theta = b \cos t$

where $a = 0.2$ and $b \sim 1.6$

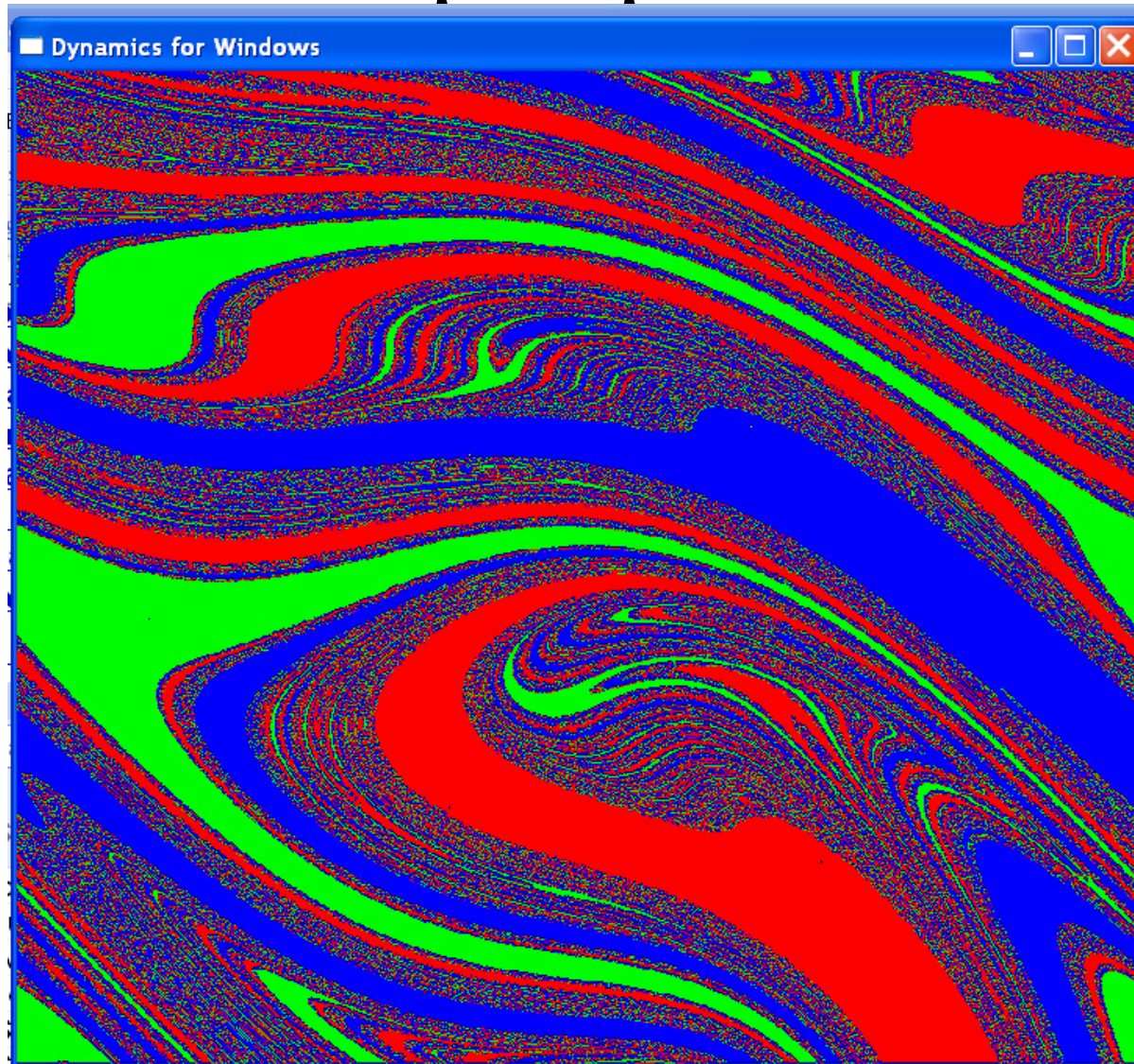
- Choose a point (θ, θ') at time 0 and find where its trajectory at times $2\pi n$ for $n = 1, 2, 3, \dots$
- The trajectory of almost every initial point goes to one of 3 or 4 fixed-point attractors, 3 if we identify the yellow and green basins.

“Wada” Boundaries of Basins

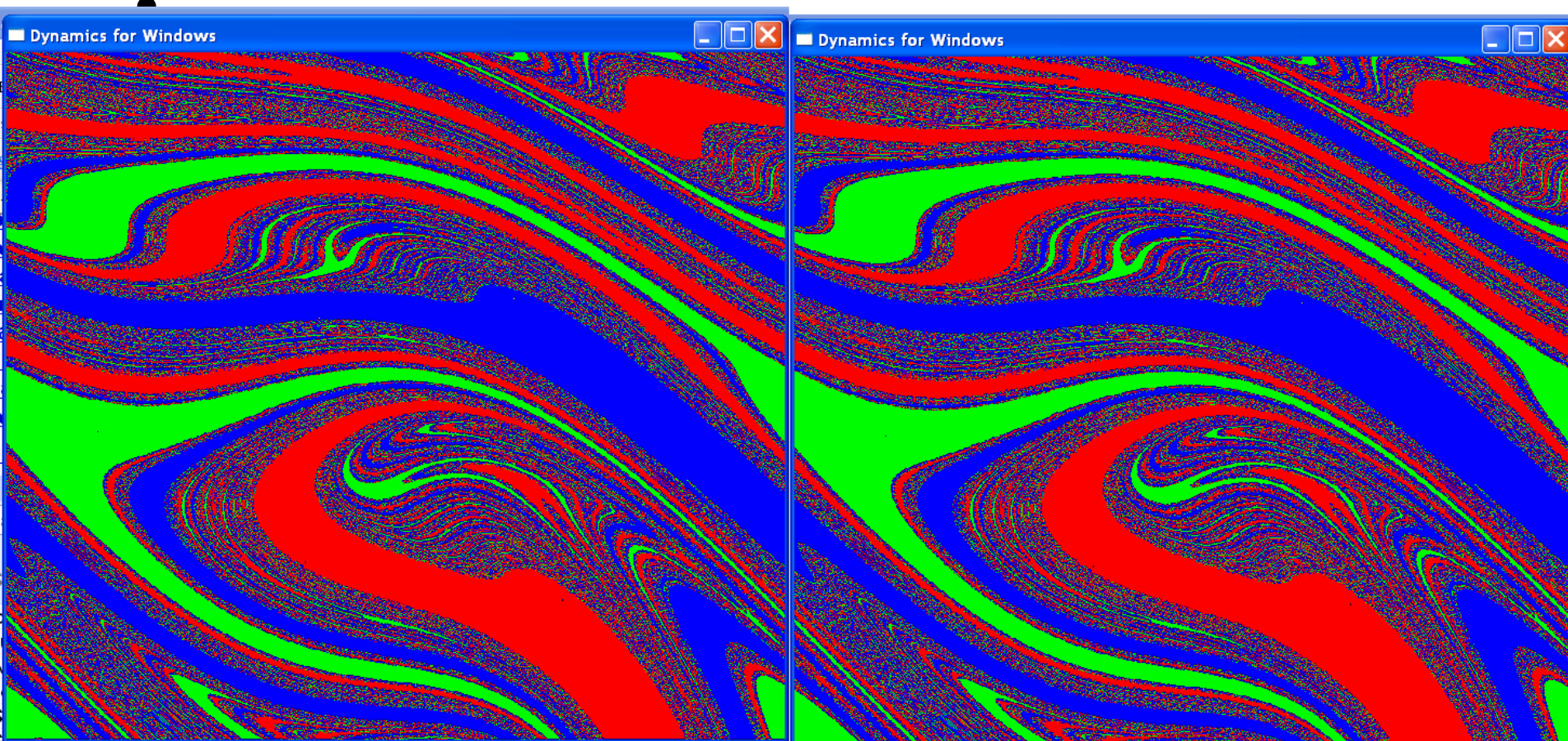
One can prove that some systems with 3 or more basins of attraction have the property that each point that is on the boundary of the basin of one is on the boundary of the basin of all the others.

- J A. Kennedy and J. A. Yorke, Basins of Wada, *Physica D* 51 (1991), 213-225.
- H. E. Nusse and J. A. Yorke, Wada basin boundaries and basin cells, *Physica D*, 90 (1996), pp. 242-261.
- H. E. Nusse and J. A. Yorke, The structure of basins of attraction and their trapping regions, *Ergodic Theory and Dyn. Sys.*, (1997), 17, pp. 463-482.
- H. E. Nusse and J. A. Yorke, Basins of attraction, *Science* (1996), 271, pp. 1376-1380.

Forced damped pendulum basins

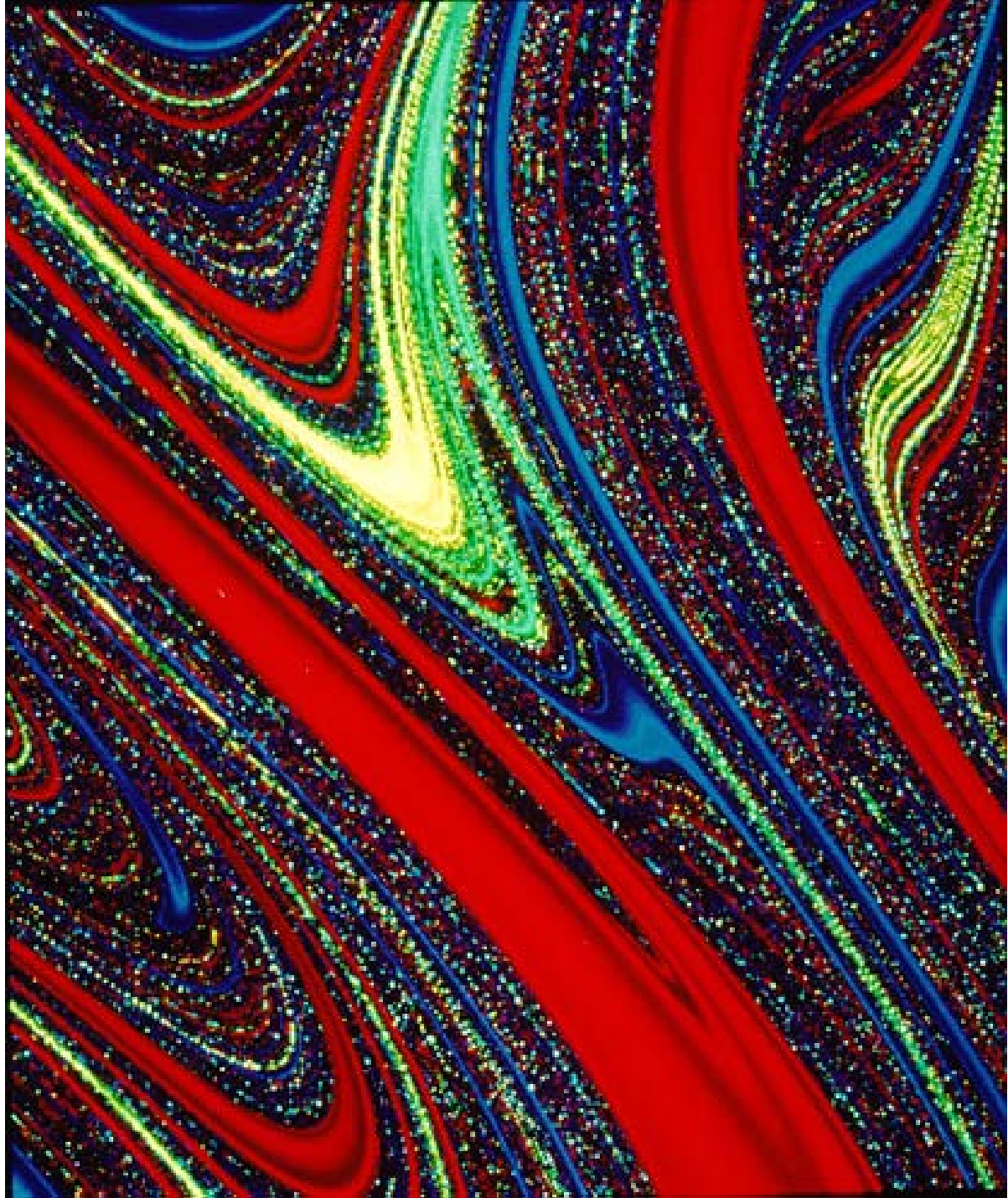


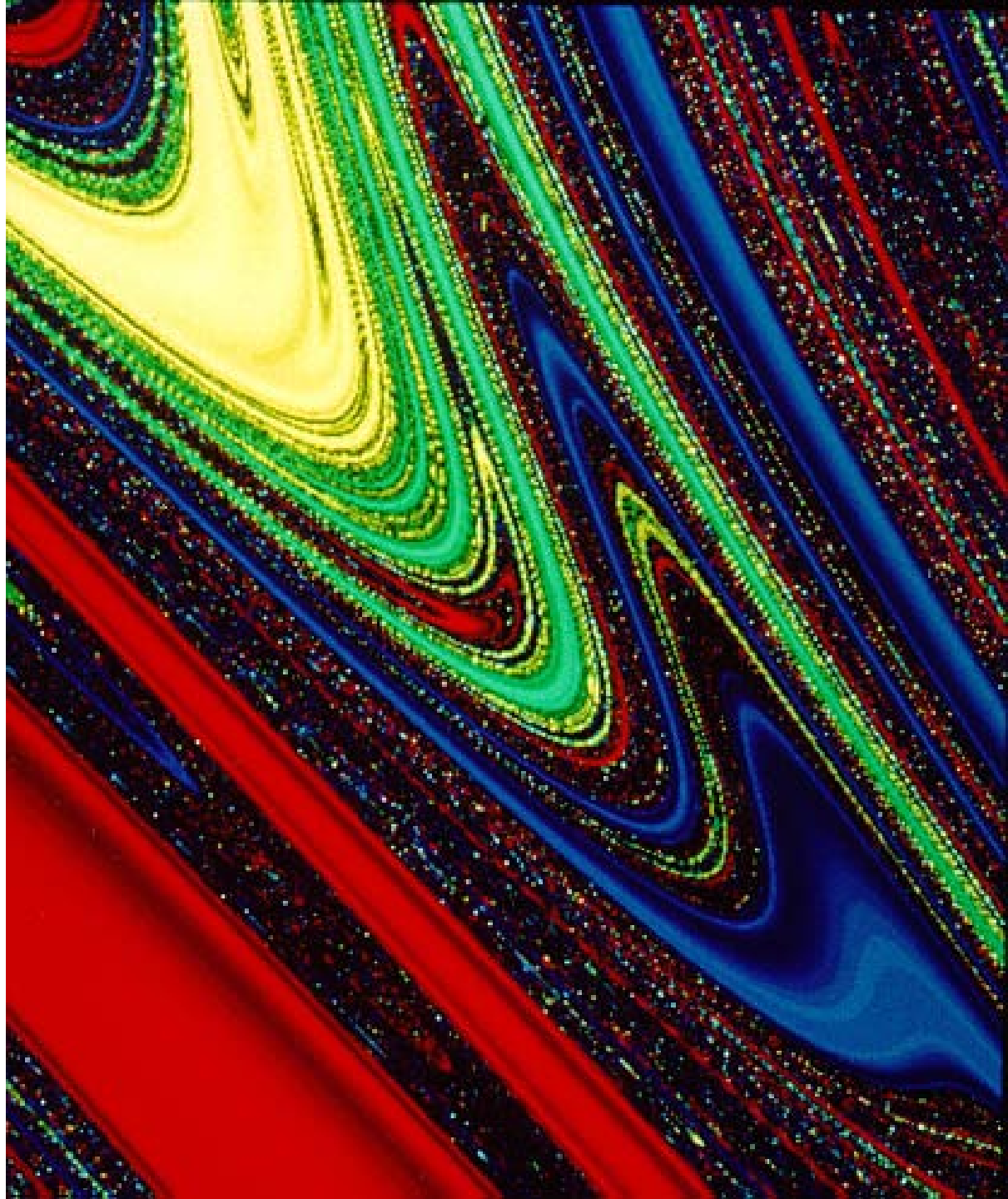
Forced damped pendulum basins

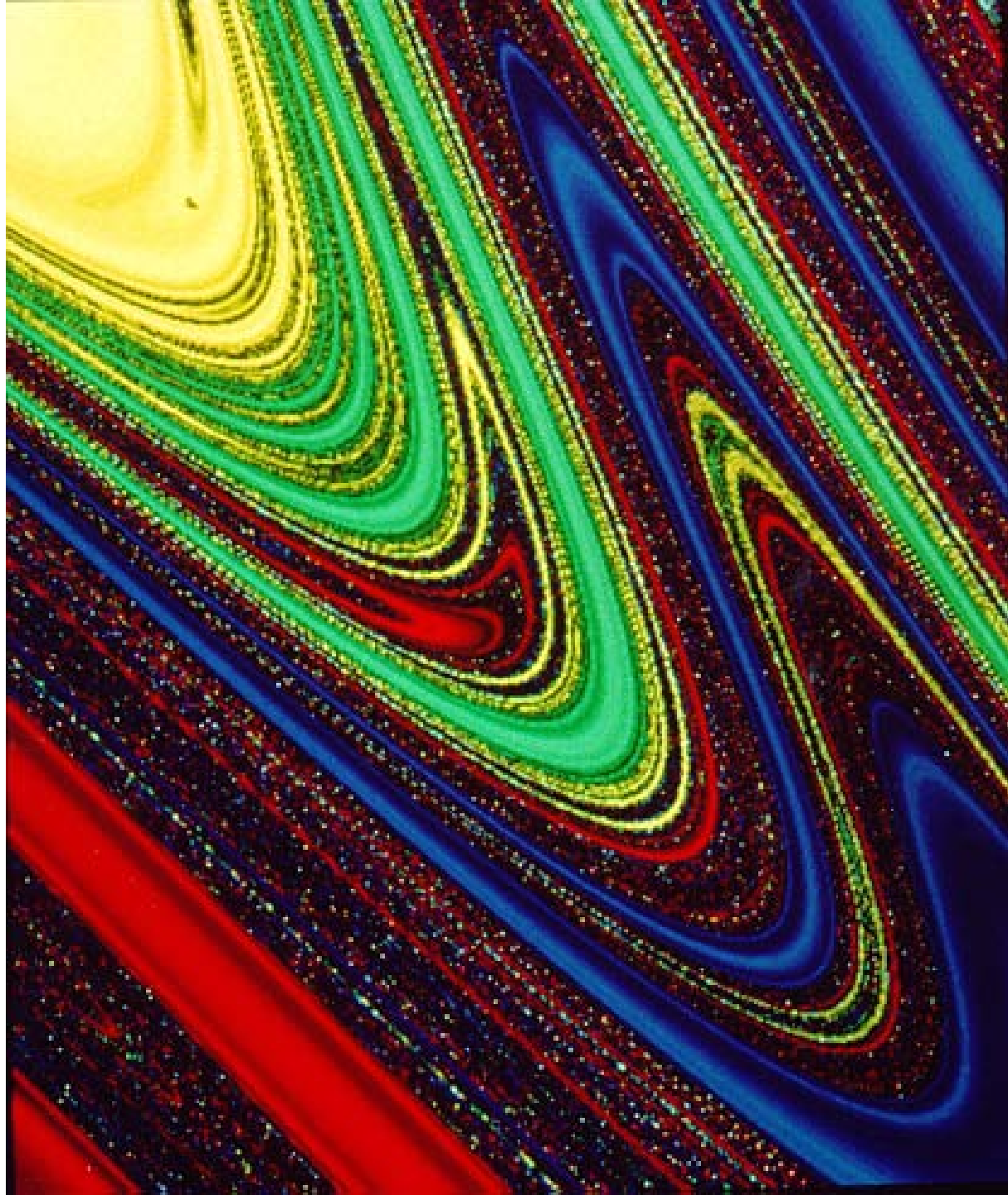


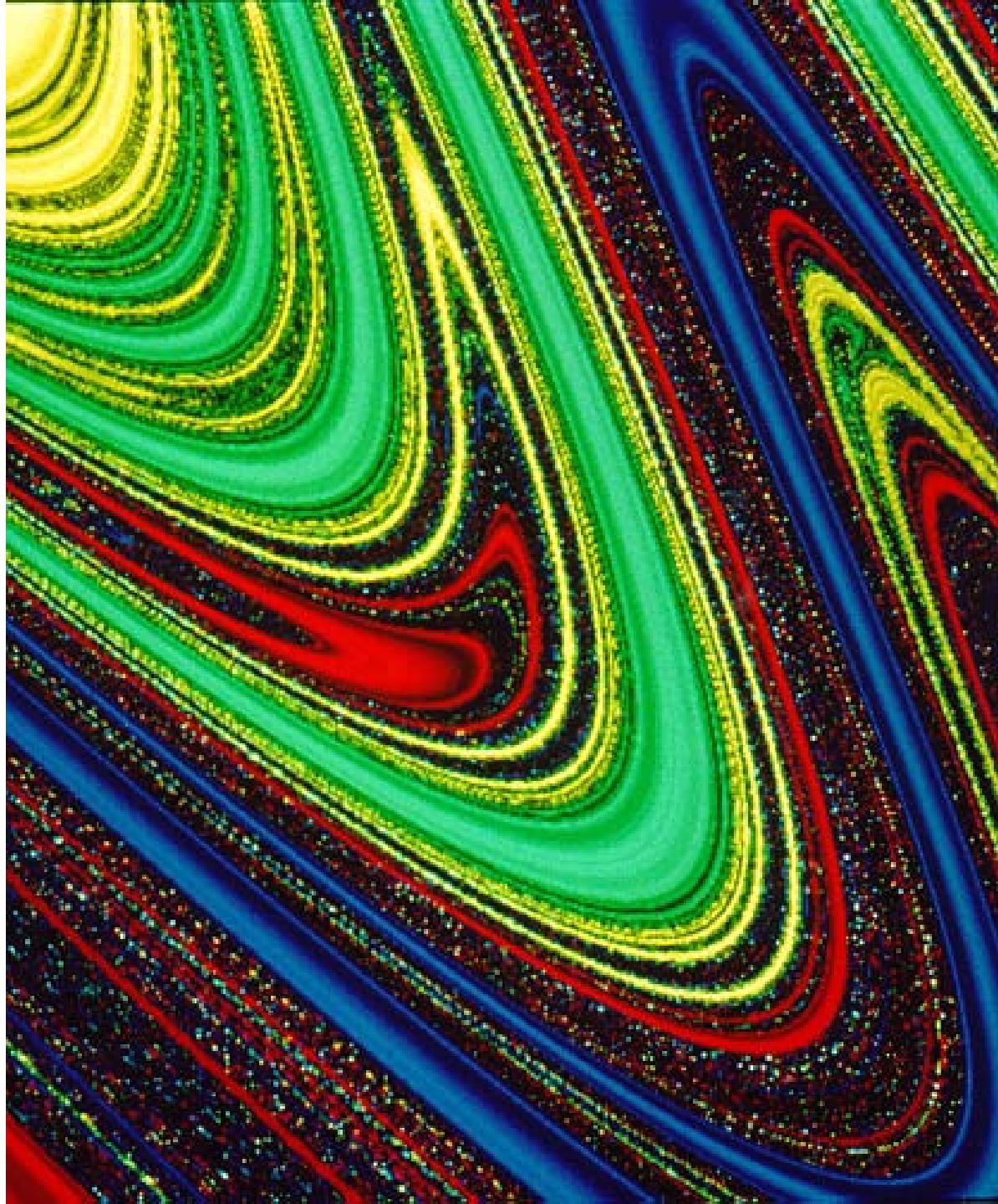
Basins of
attraction
Forced
damped
pendulum

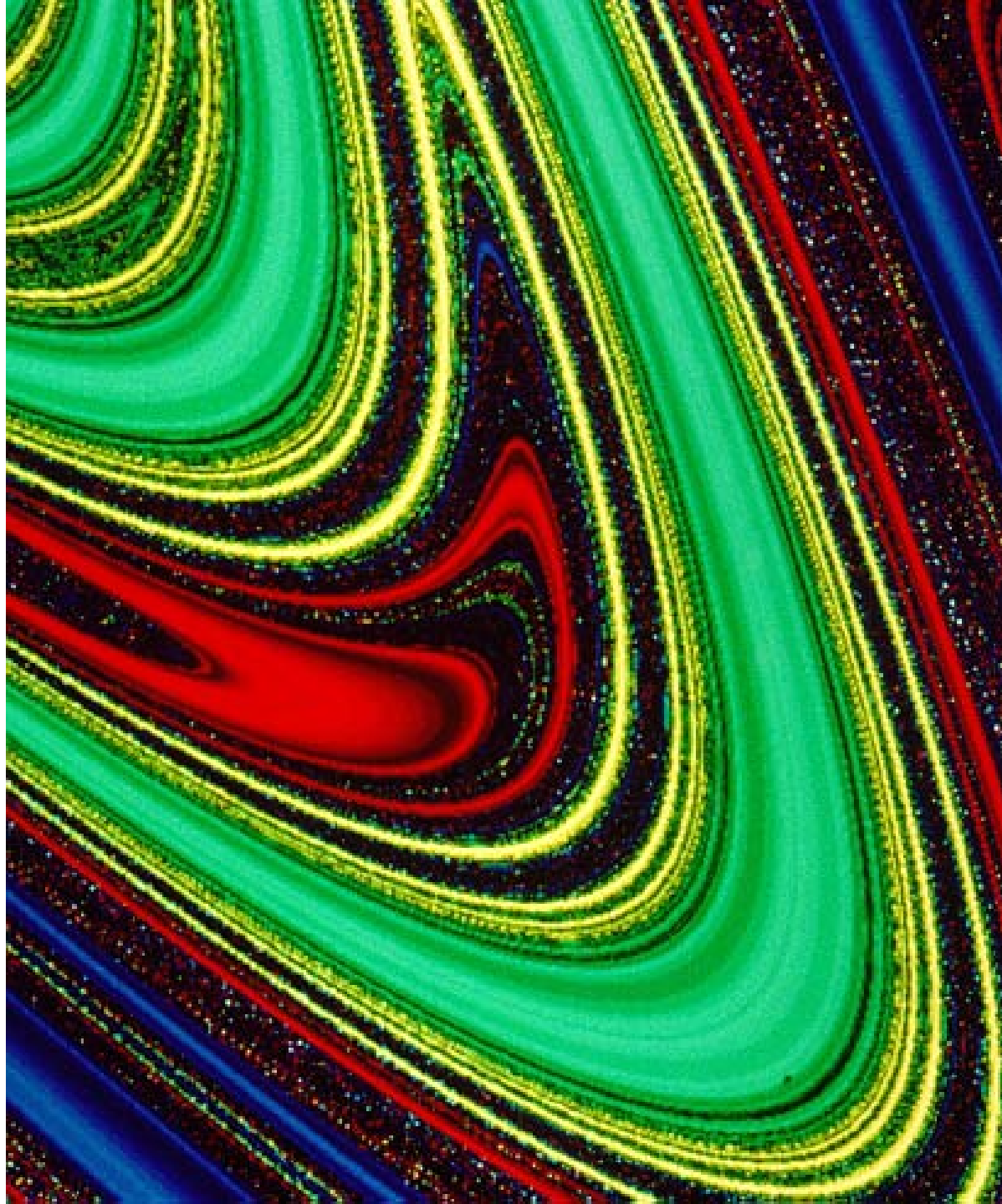
Zoom

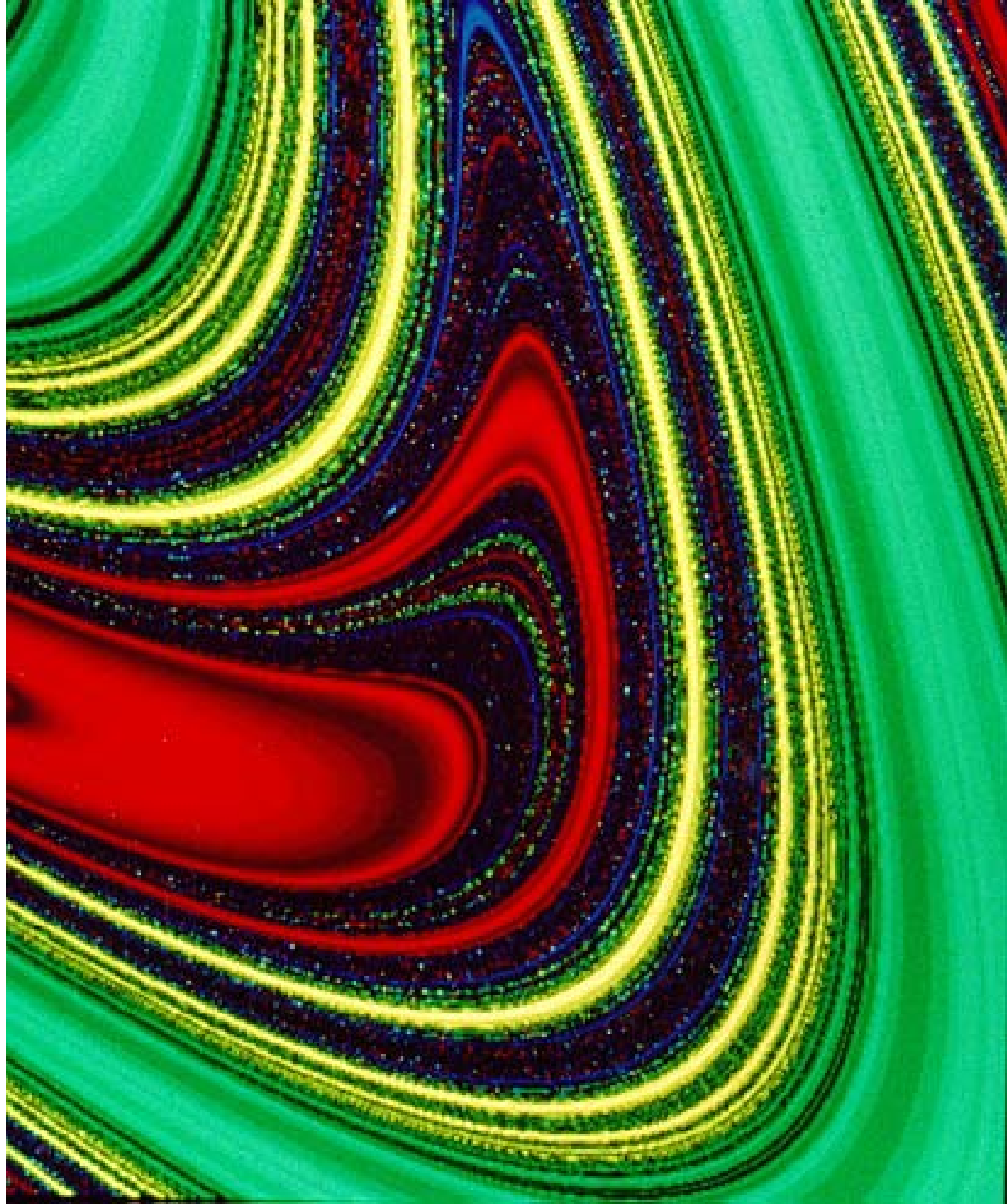


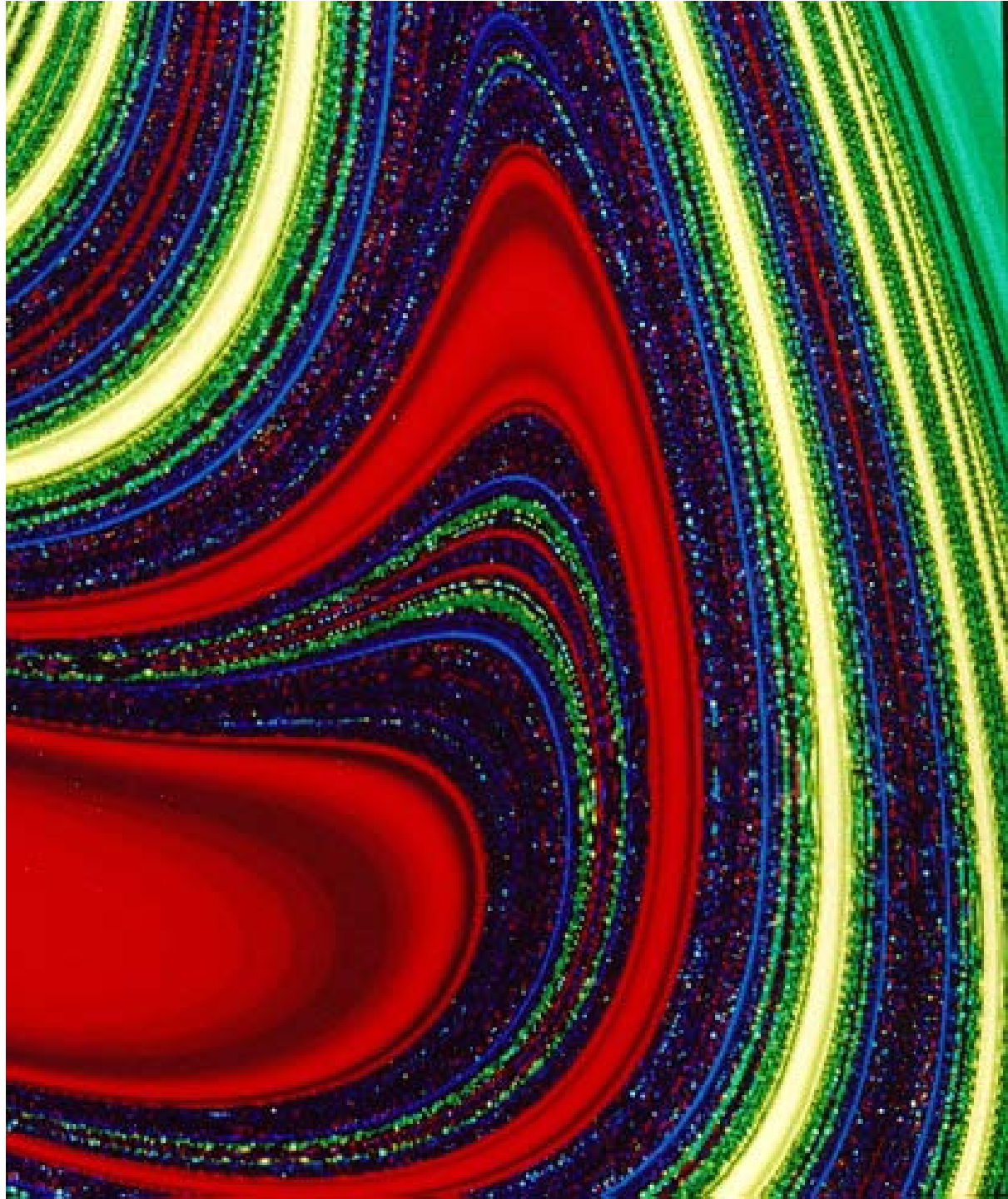


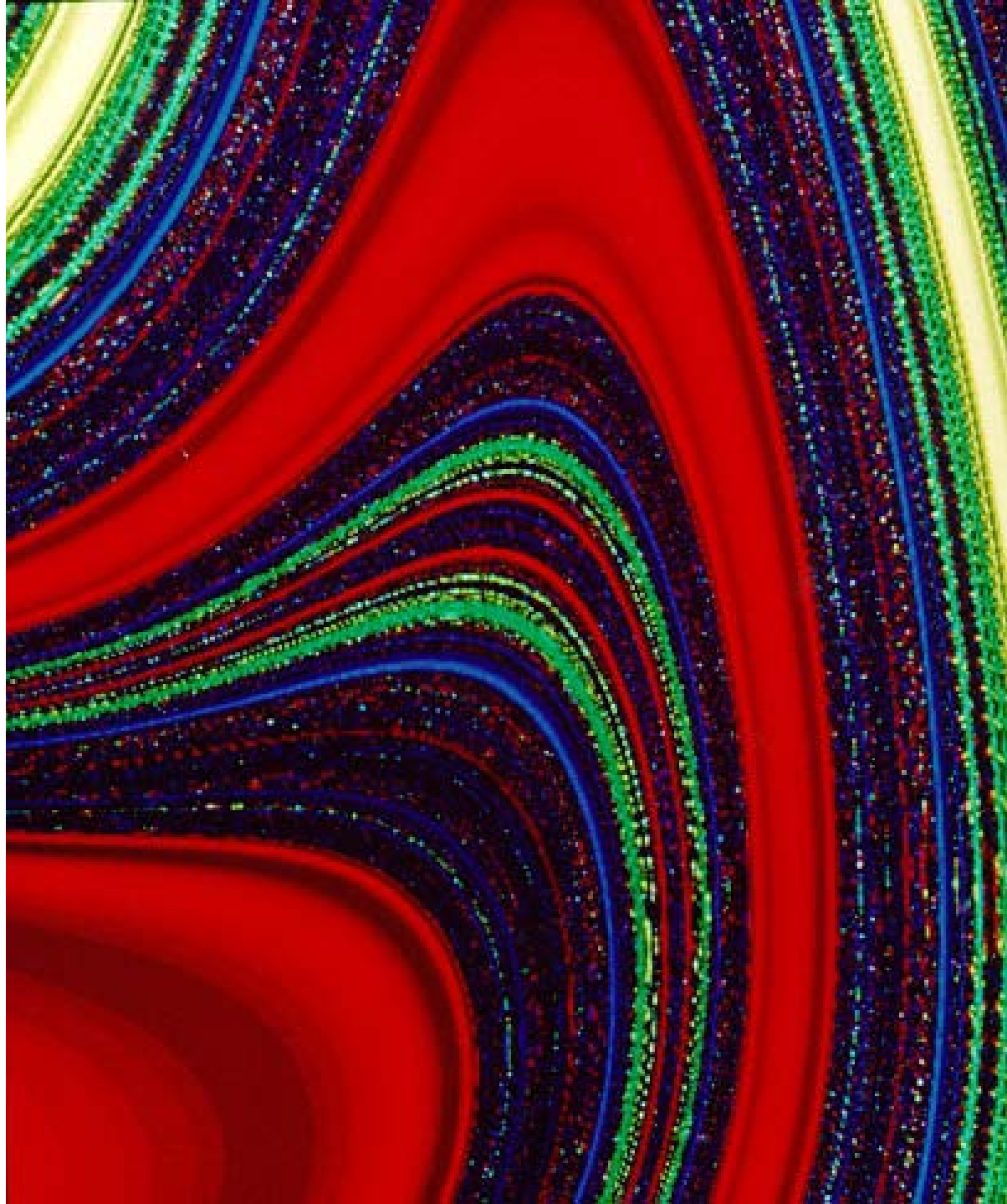


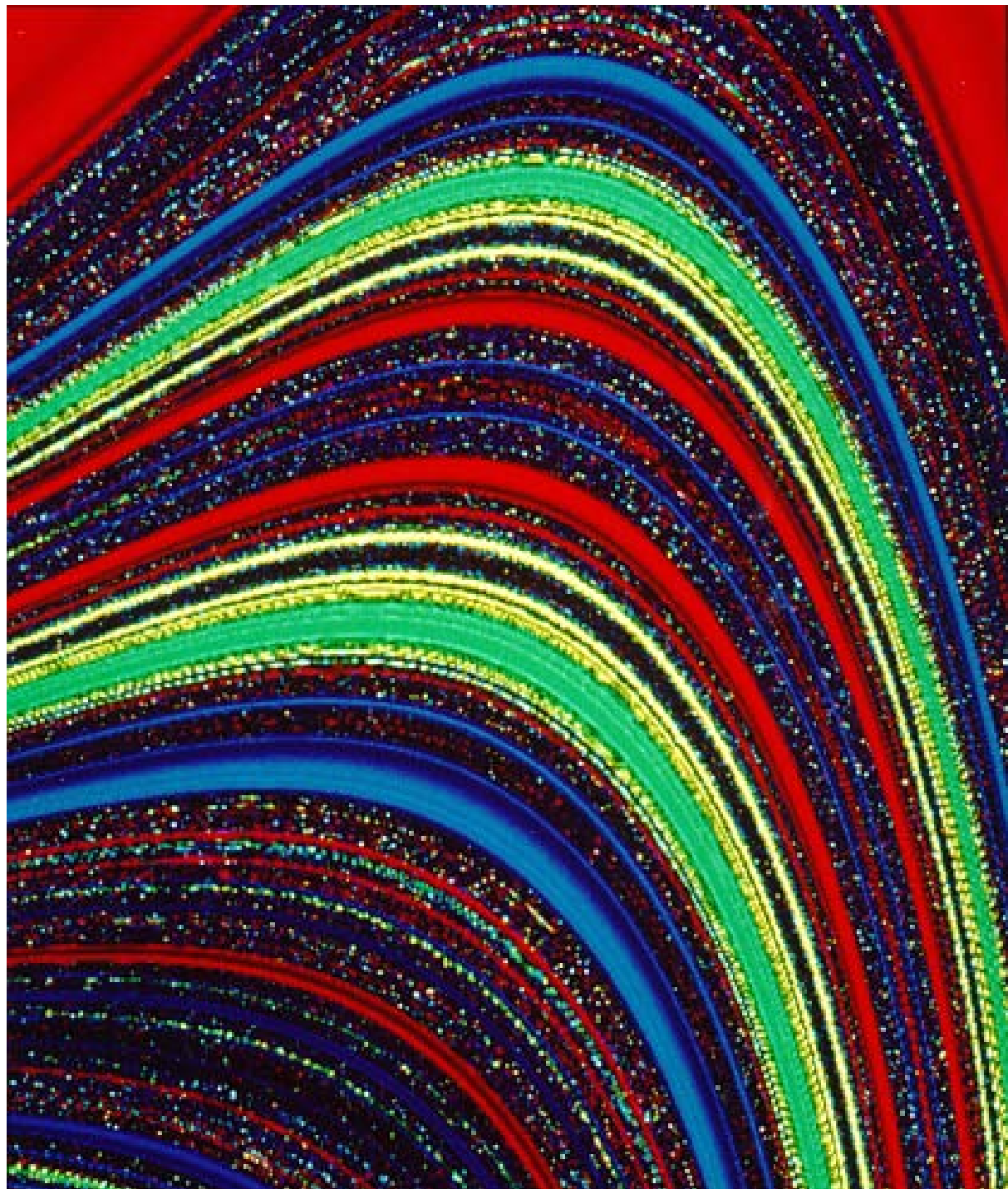


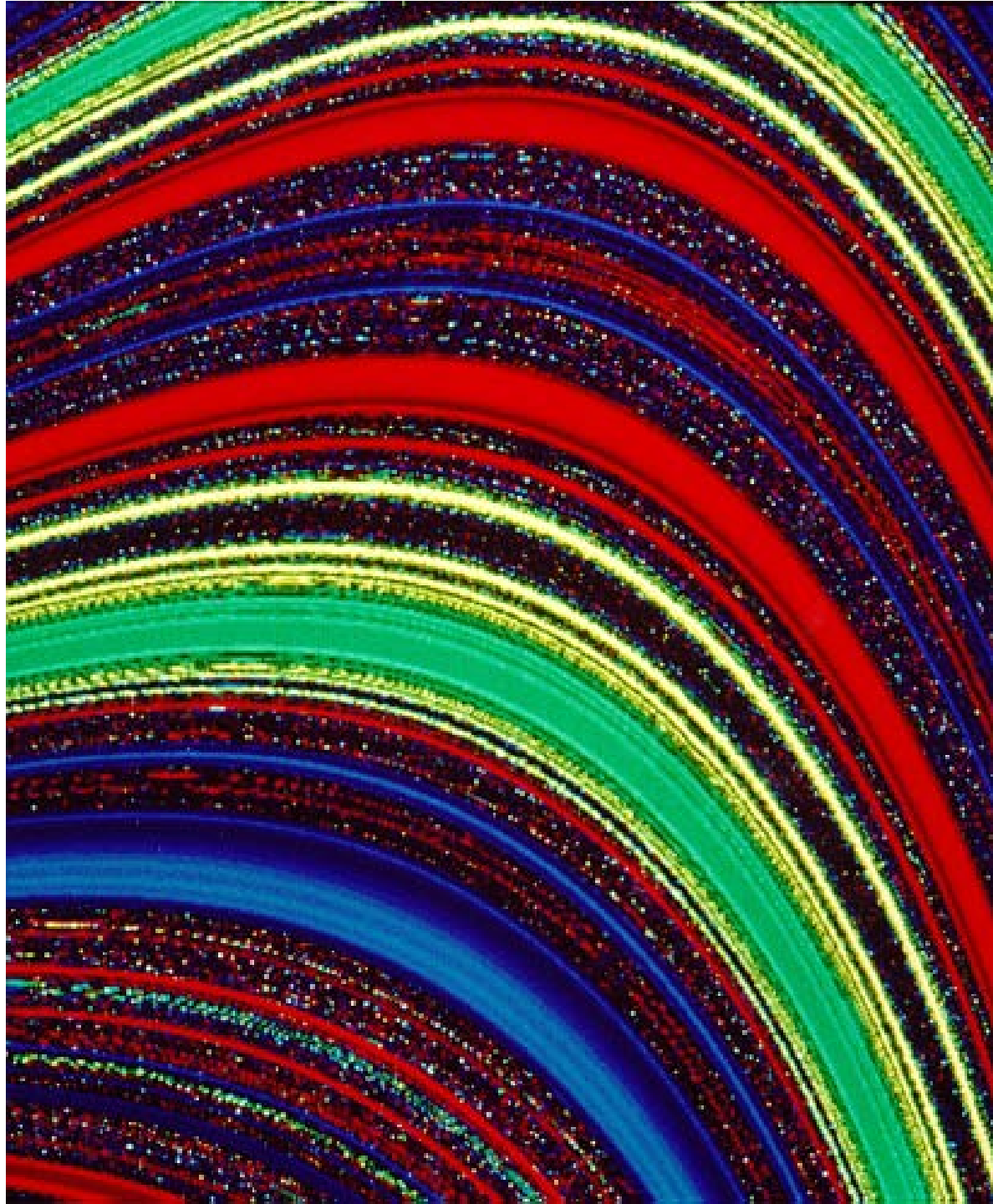


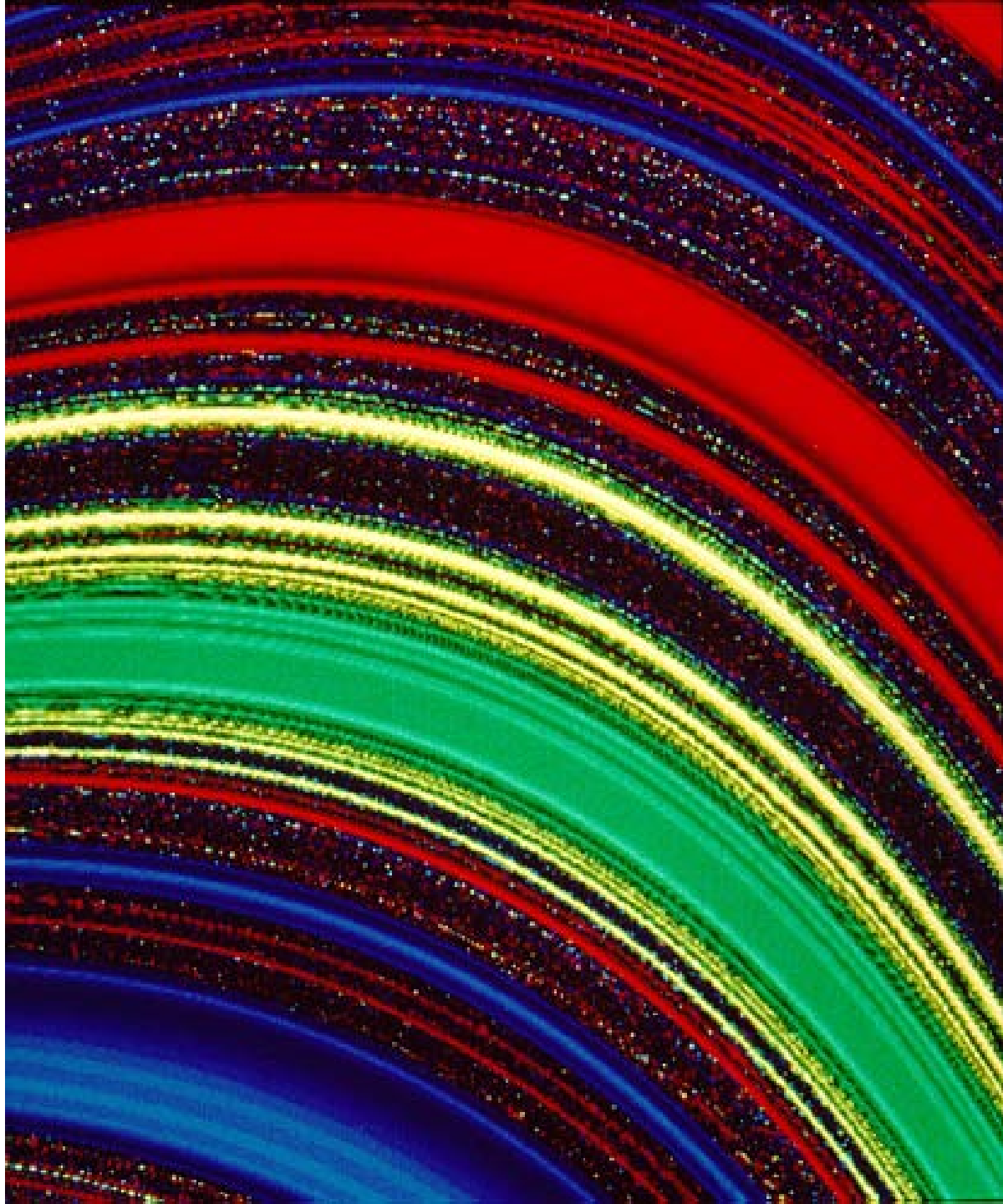


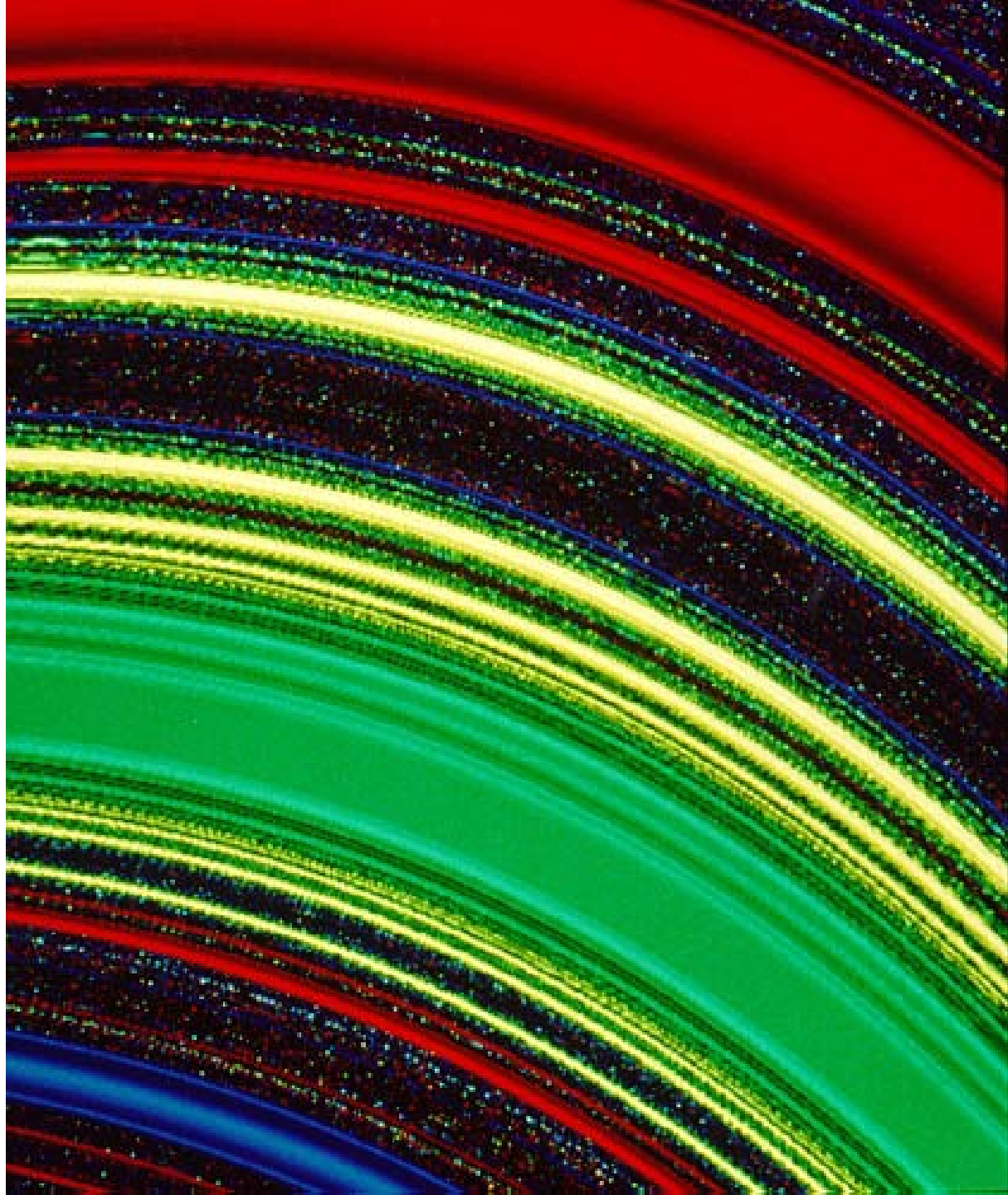


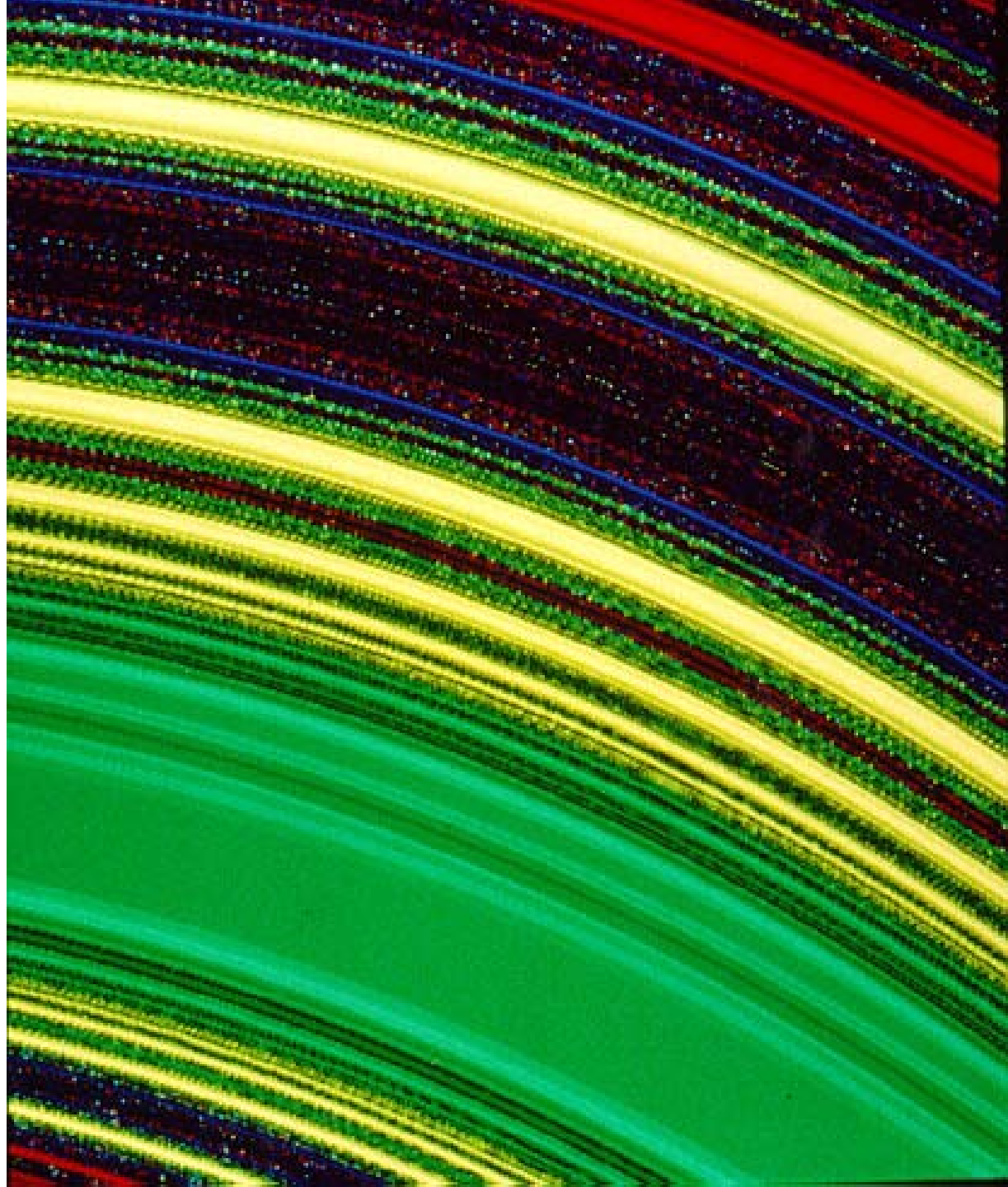


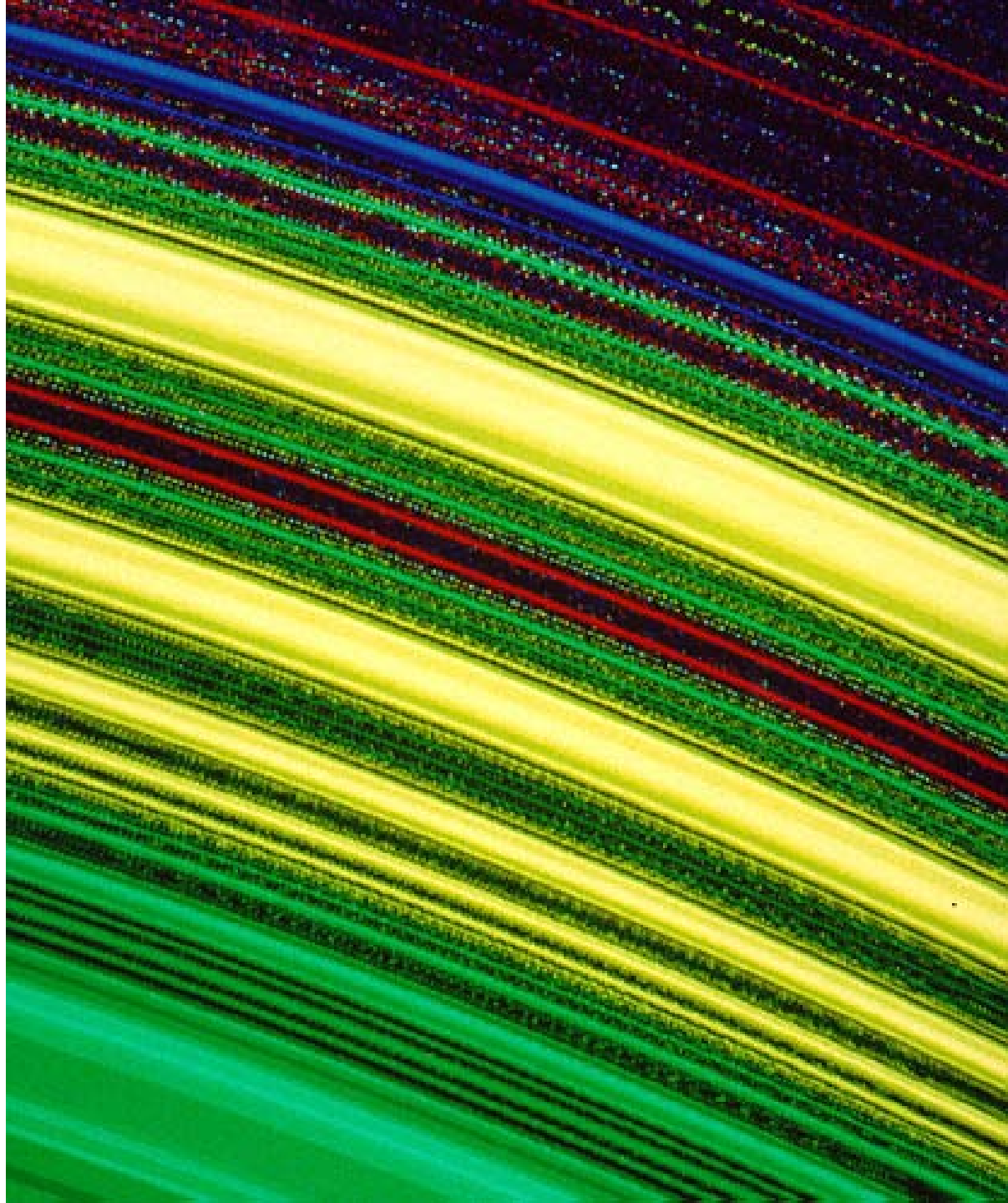


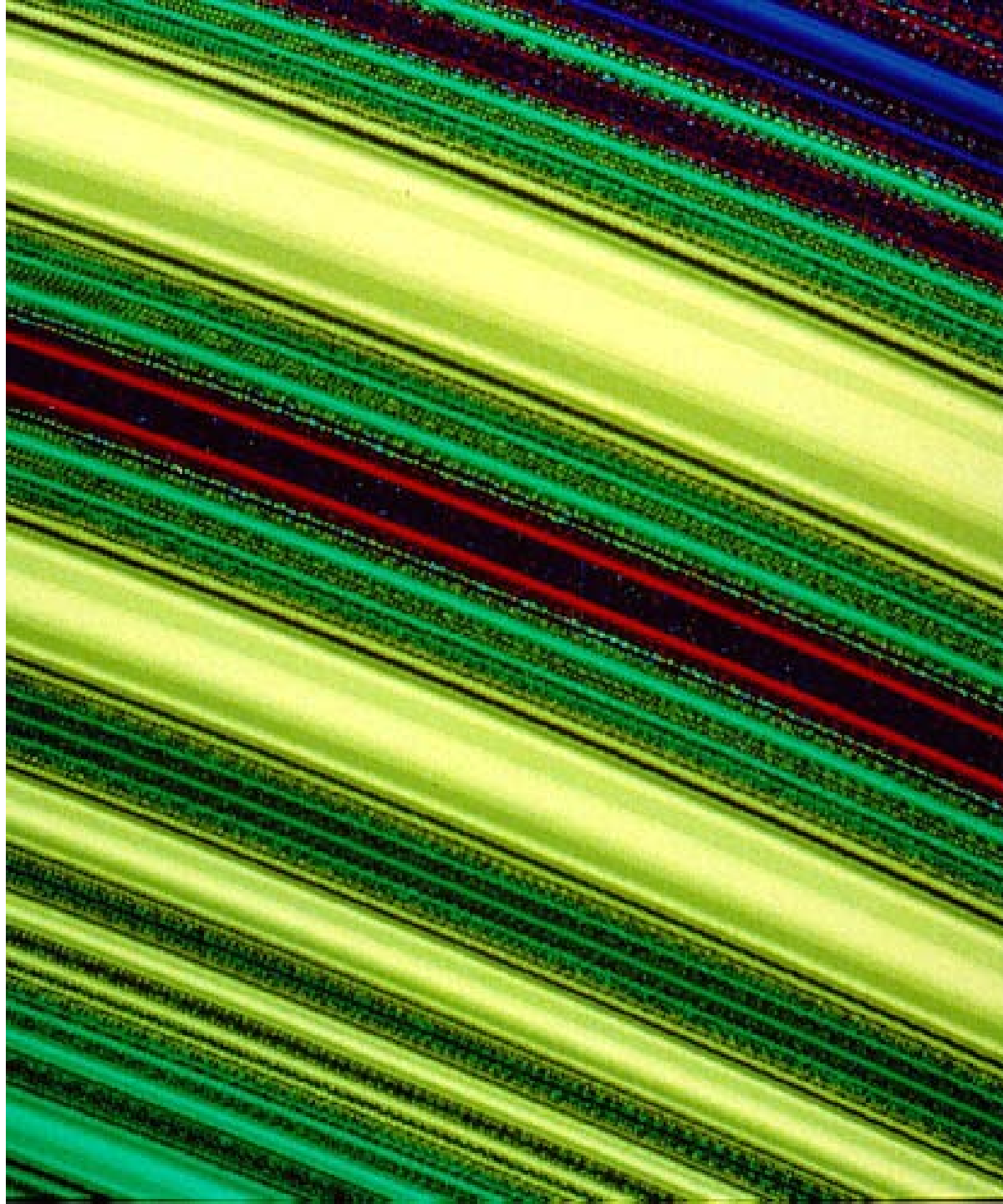




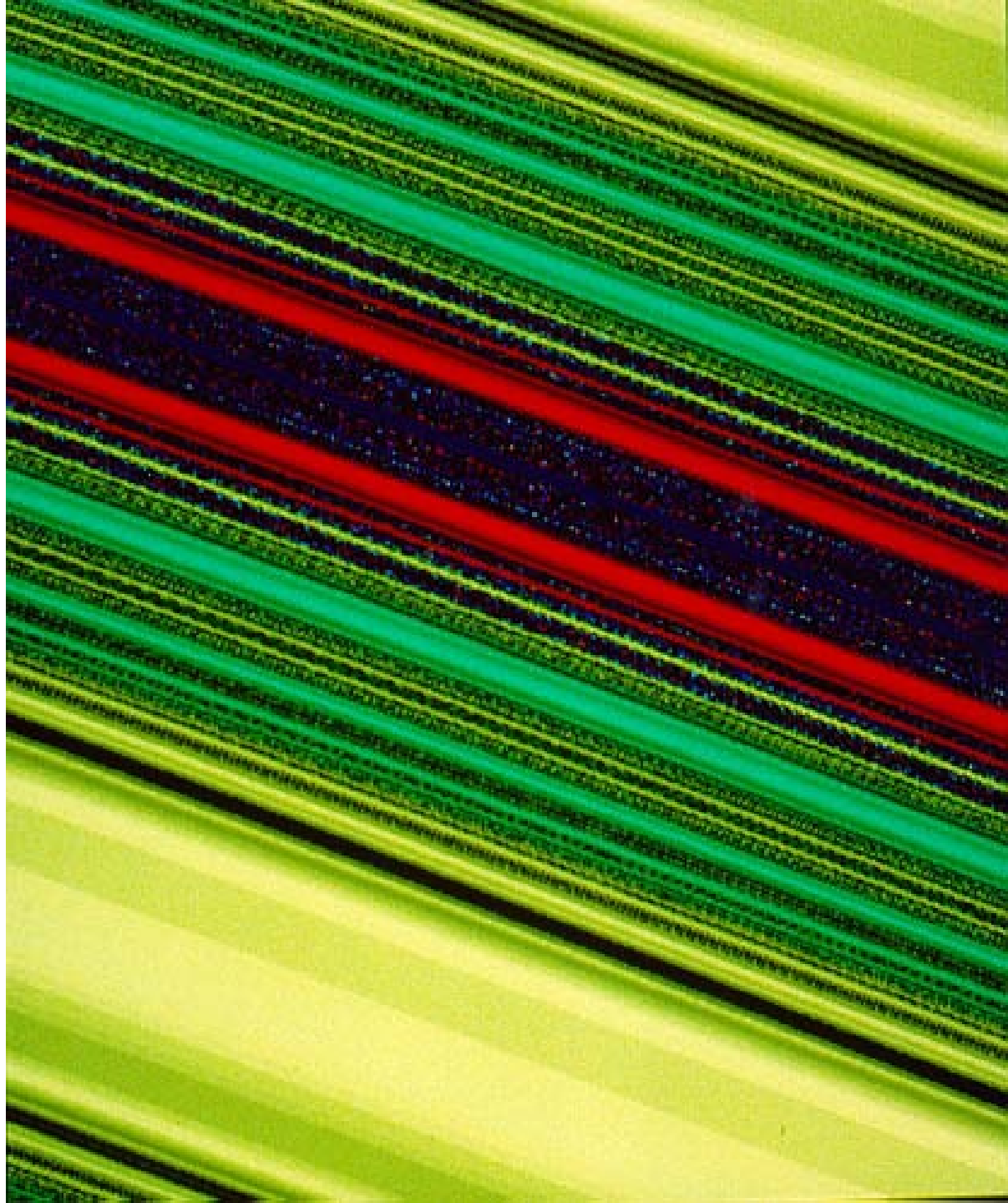


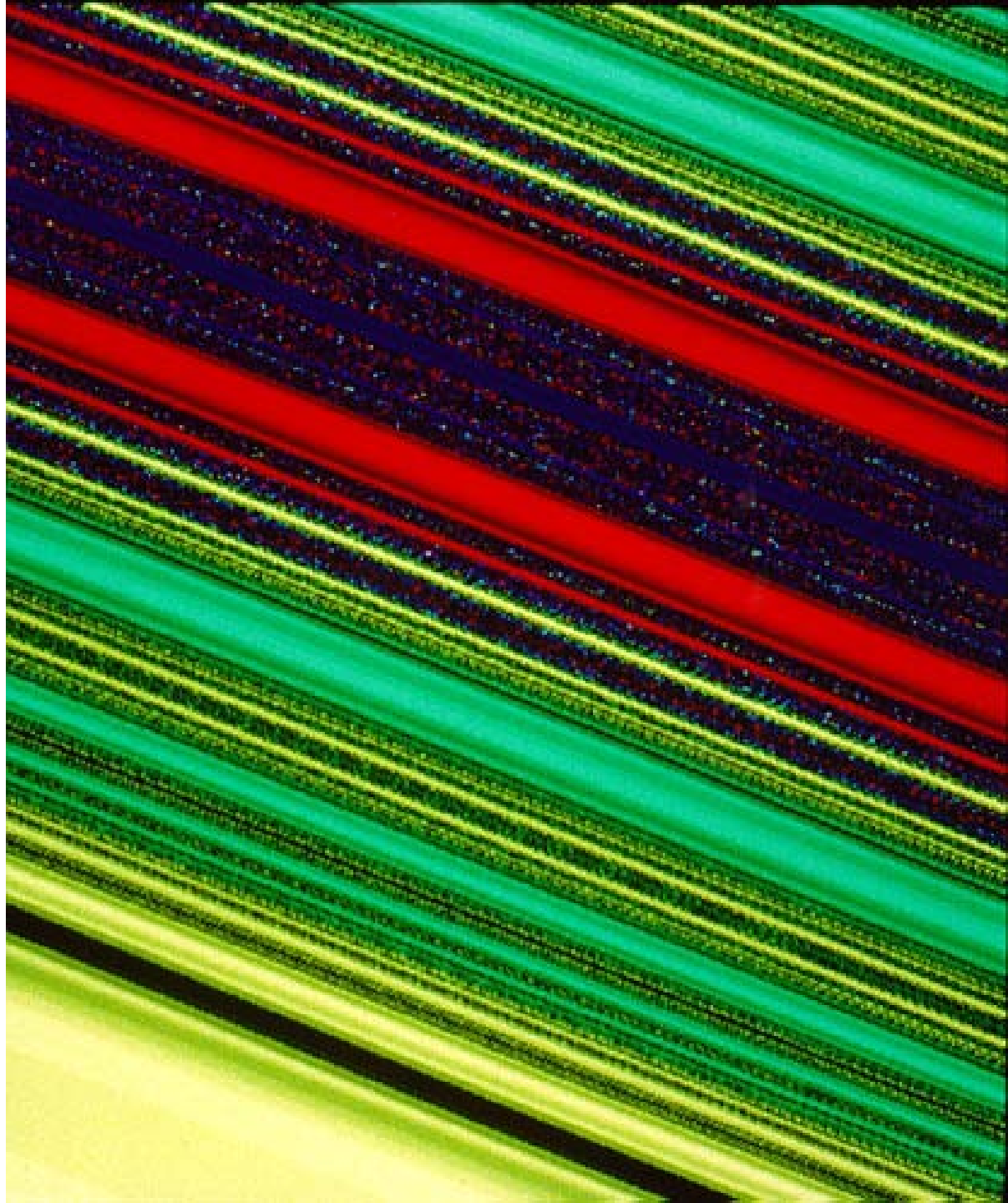


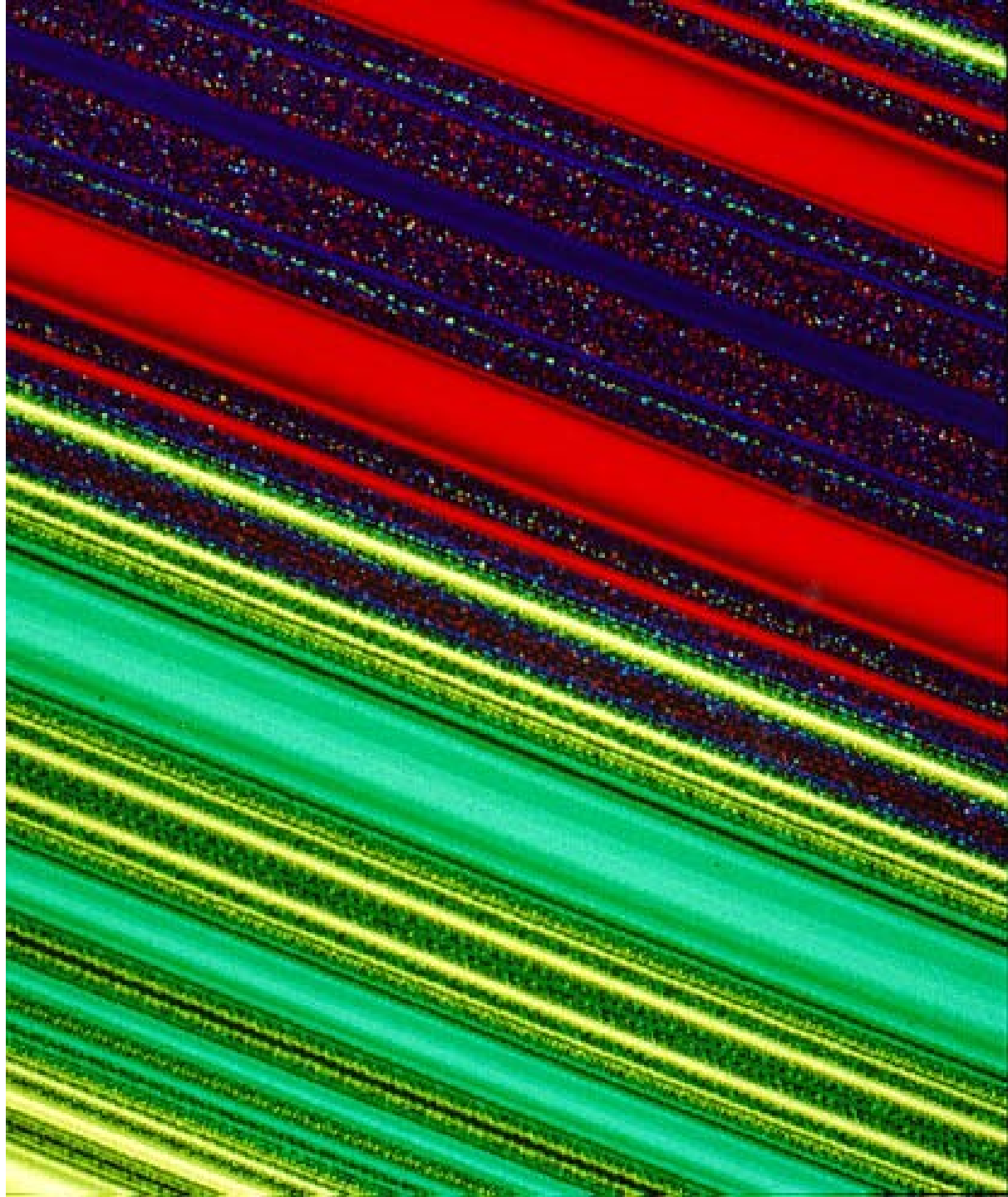


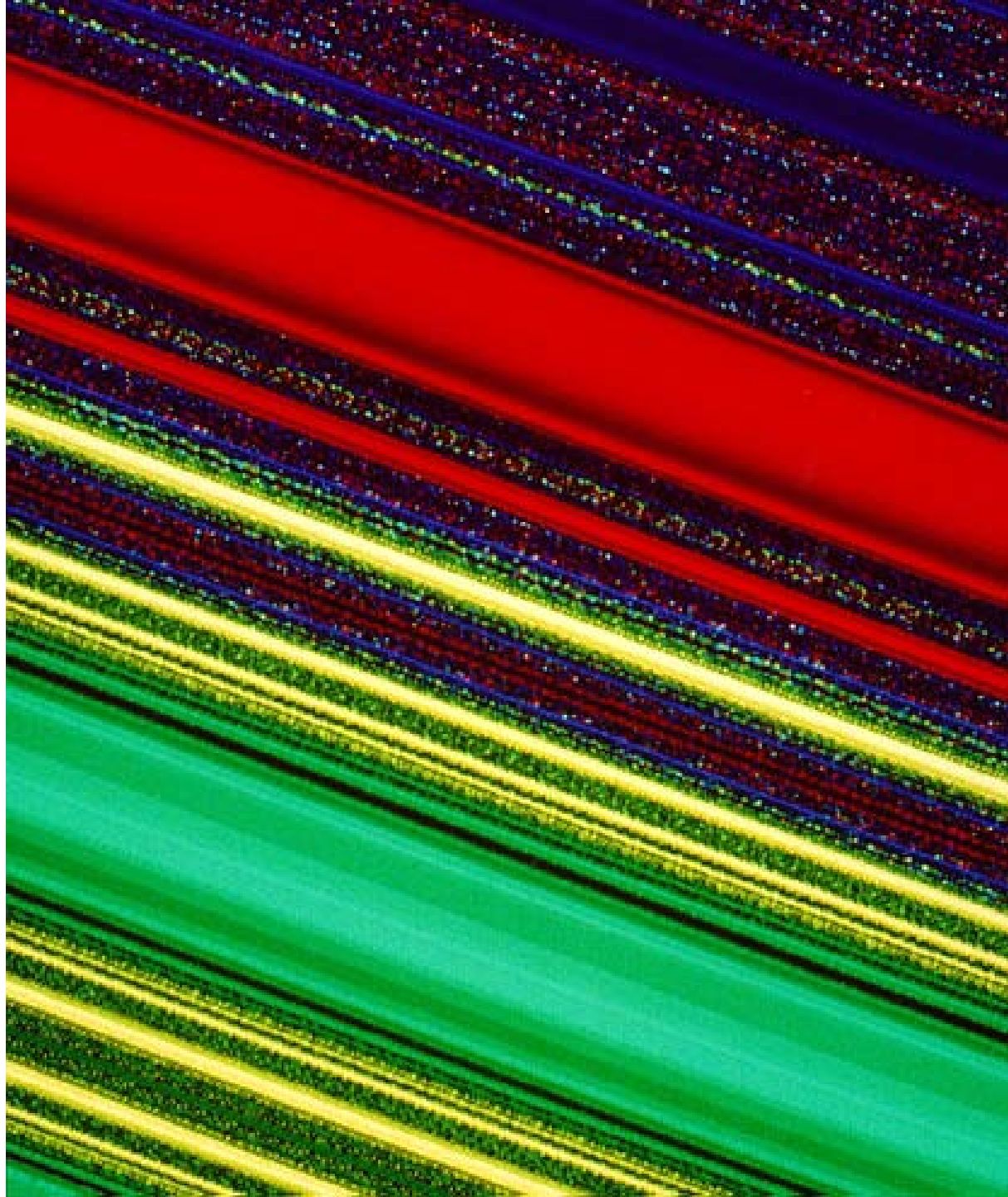


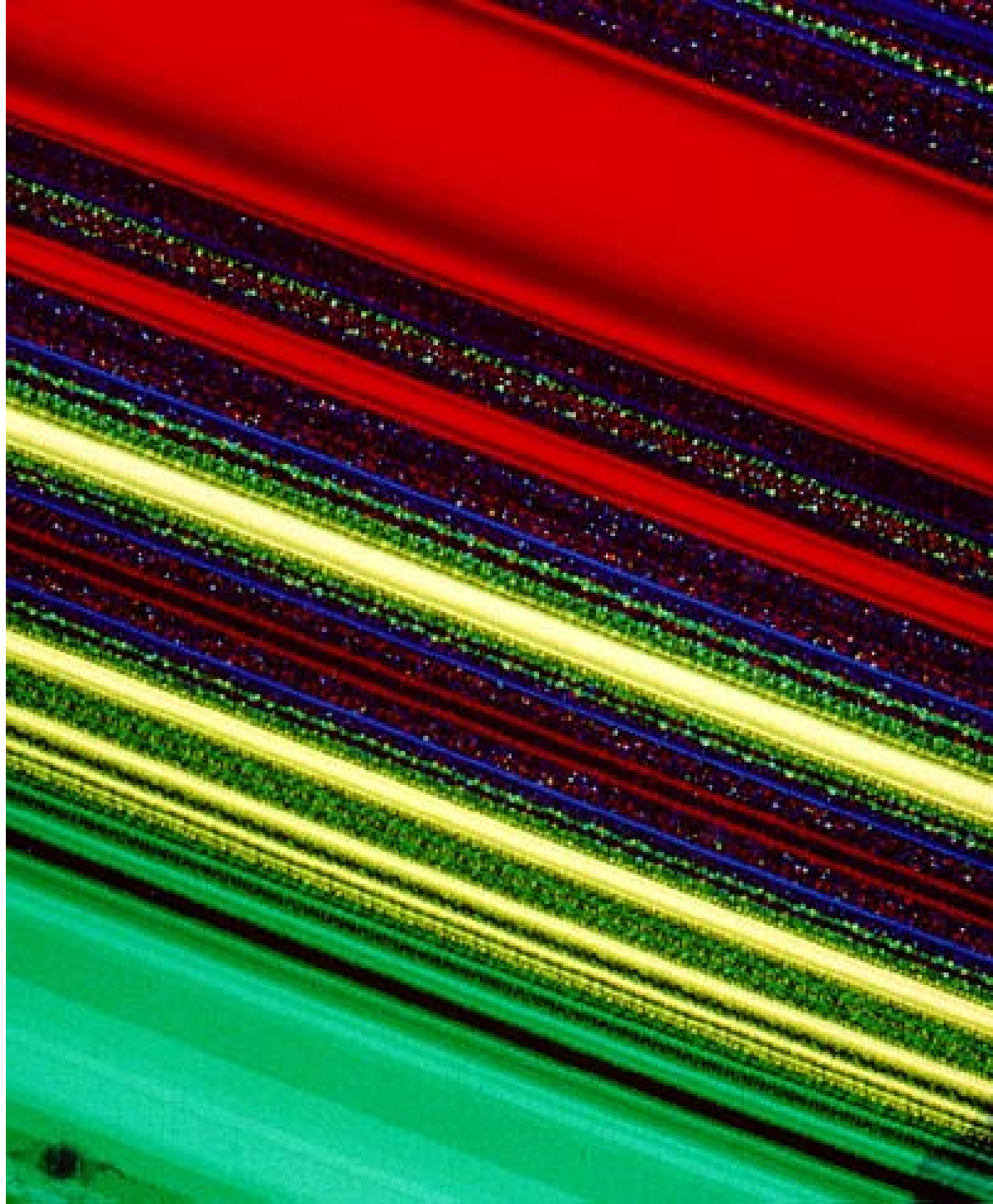








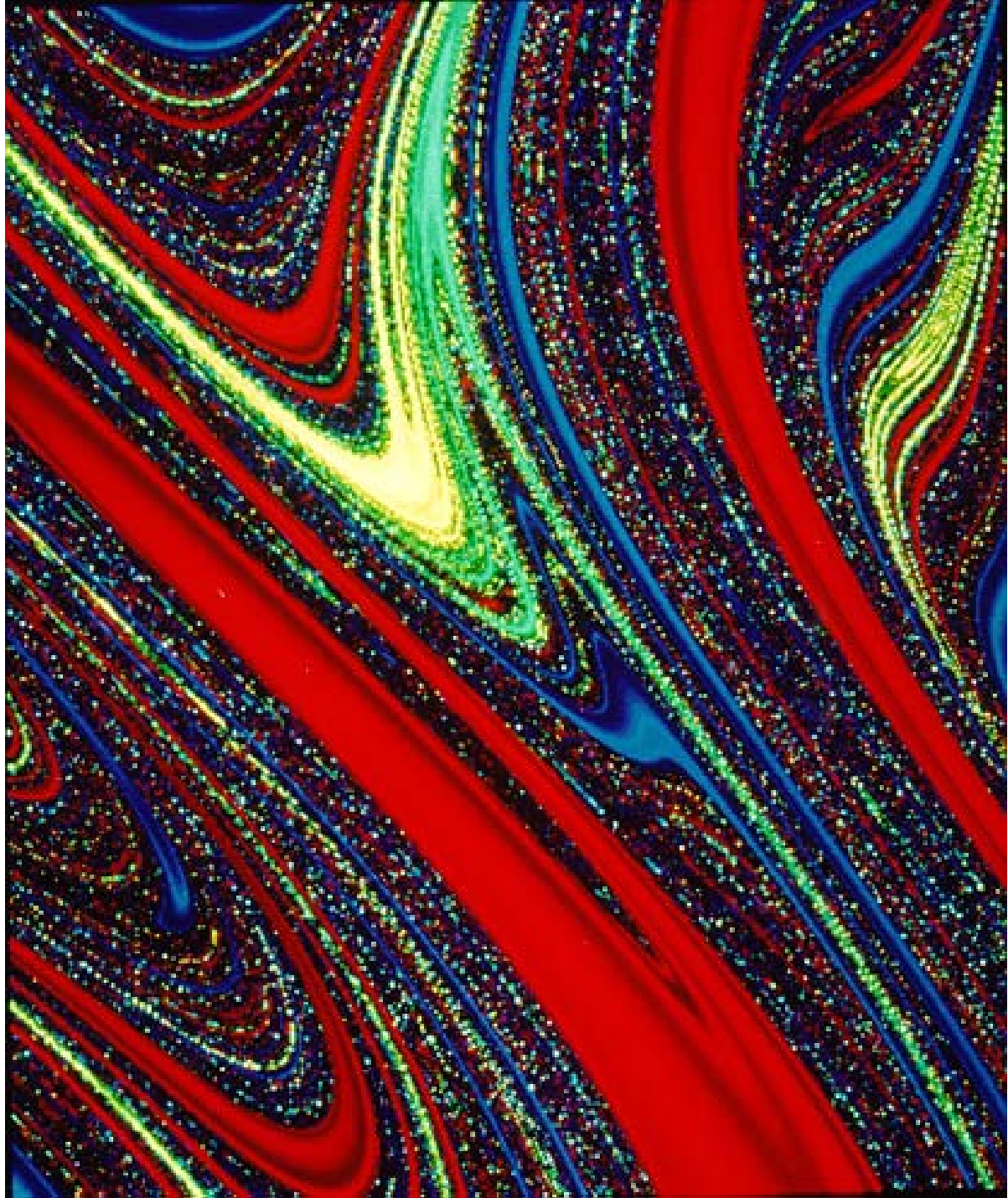


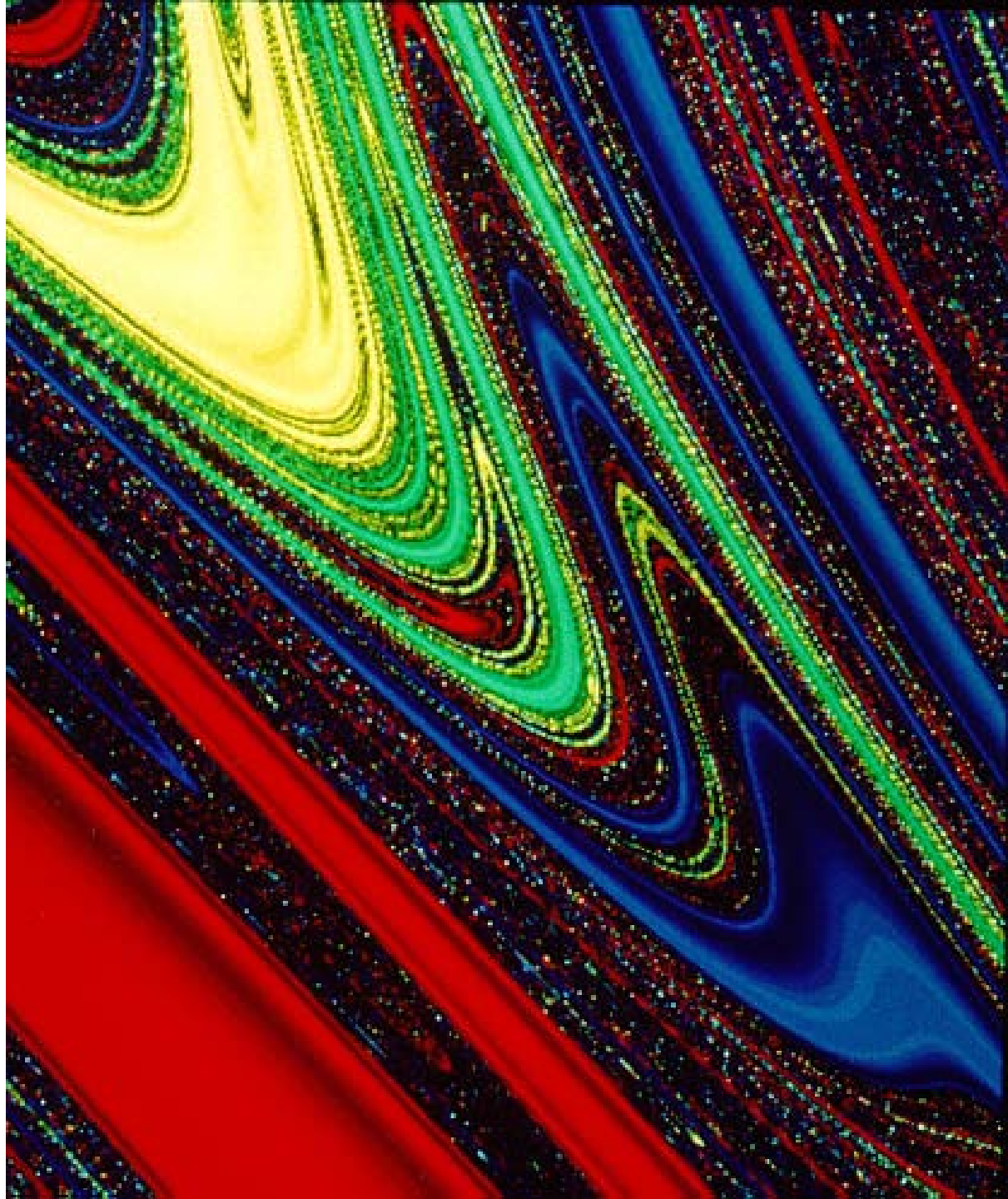


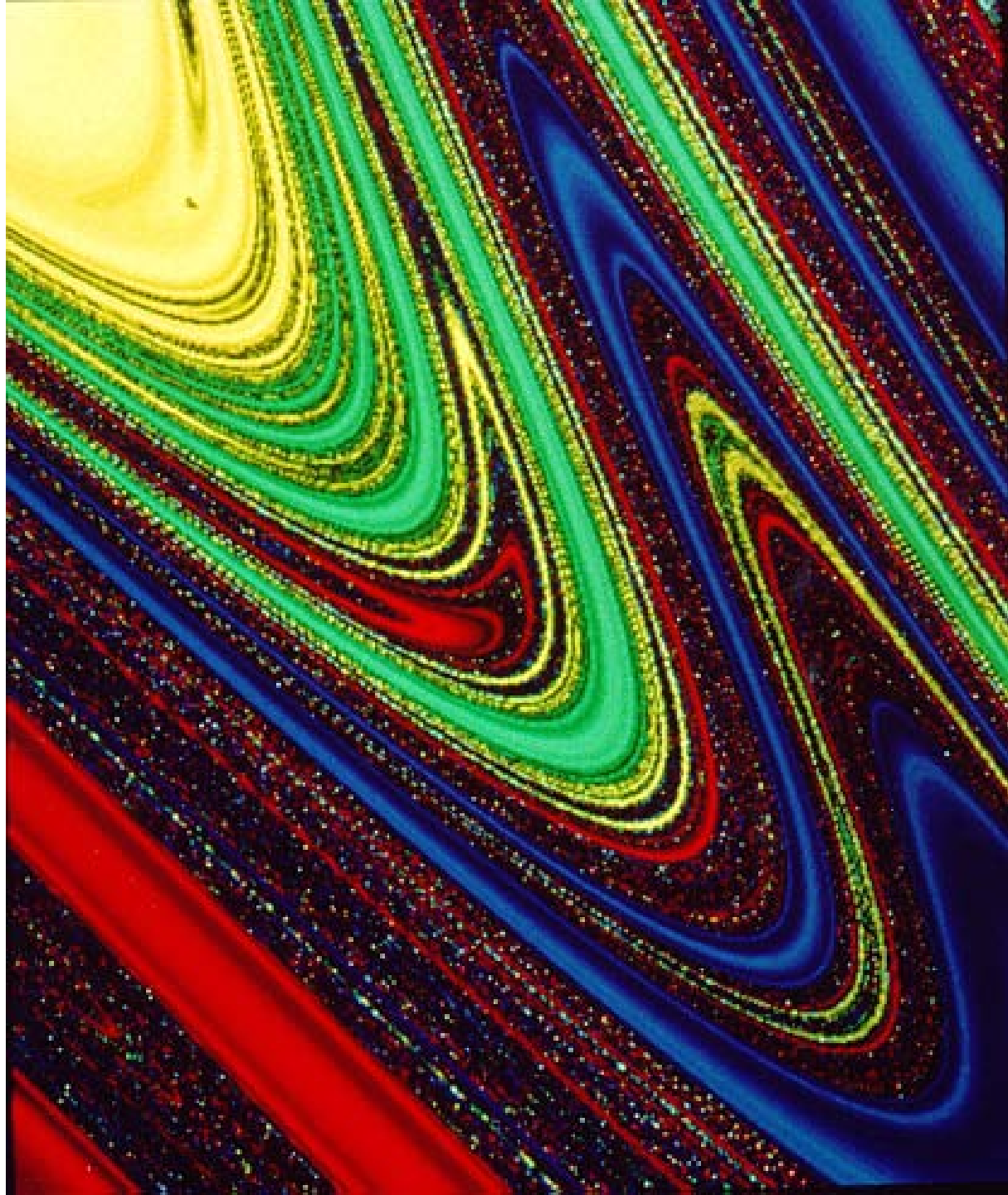
Fini
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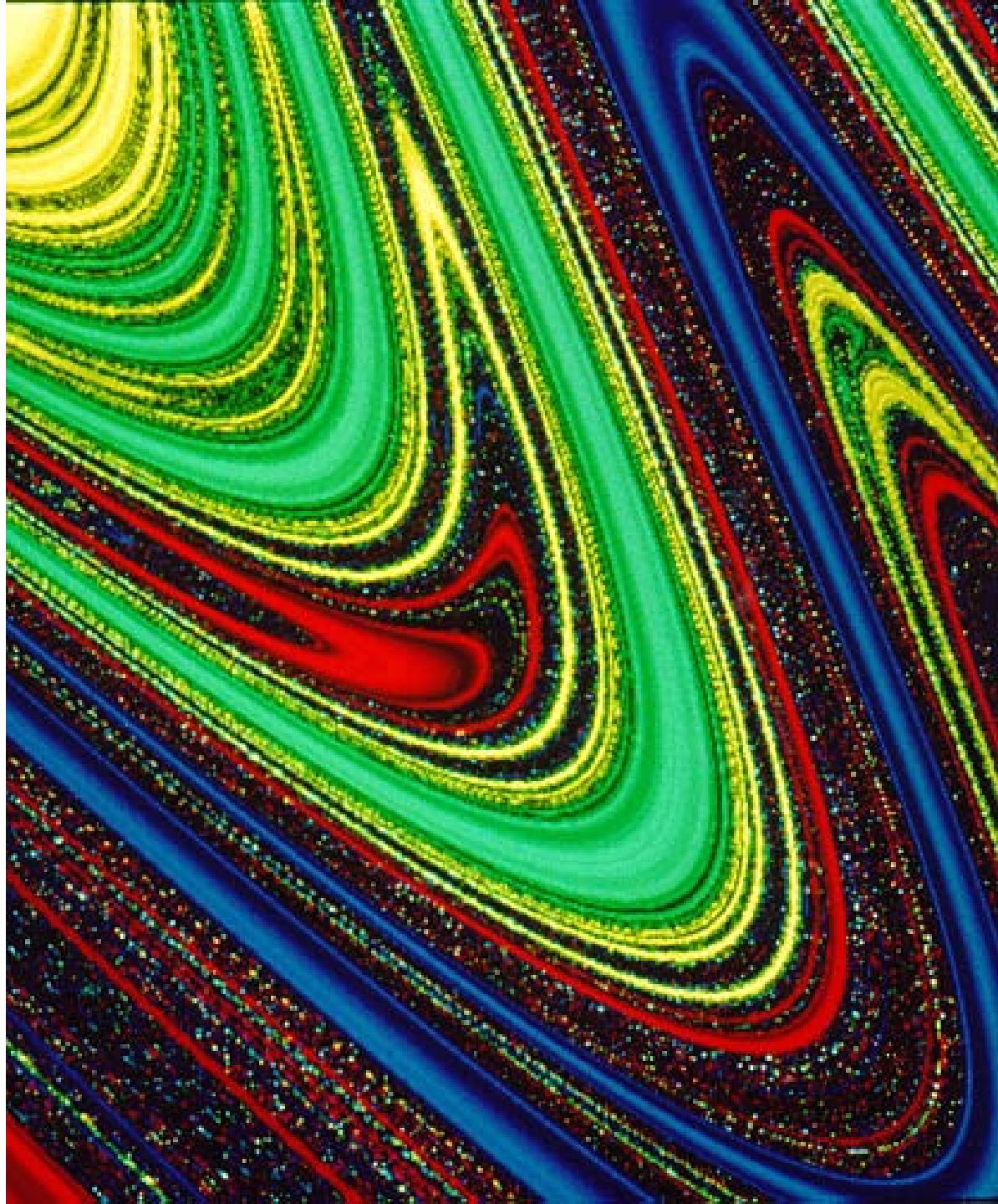
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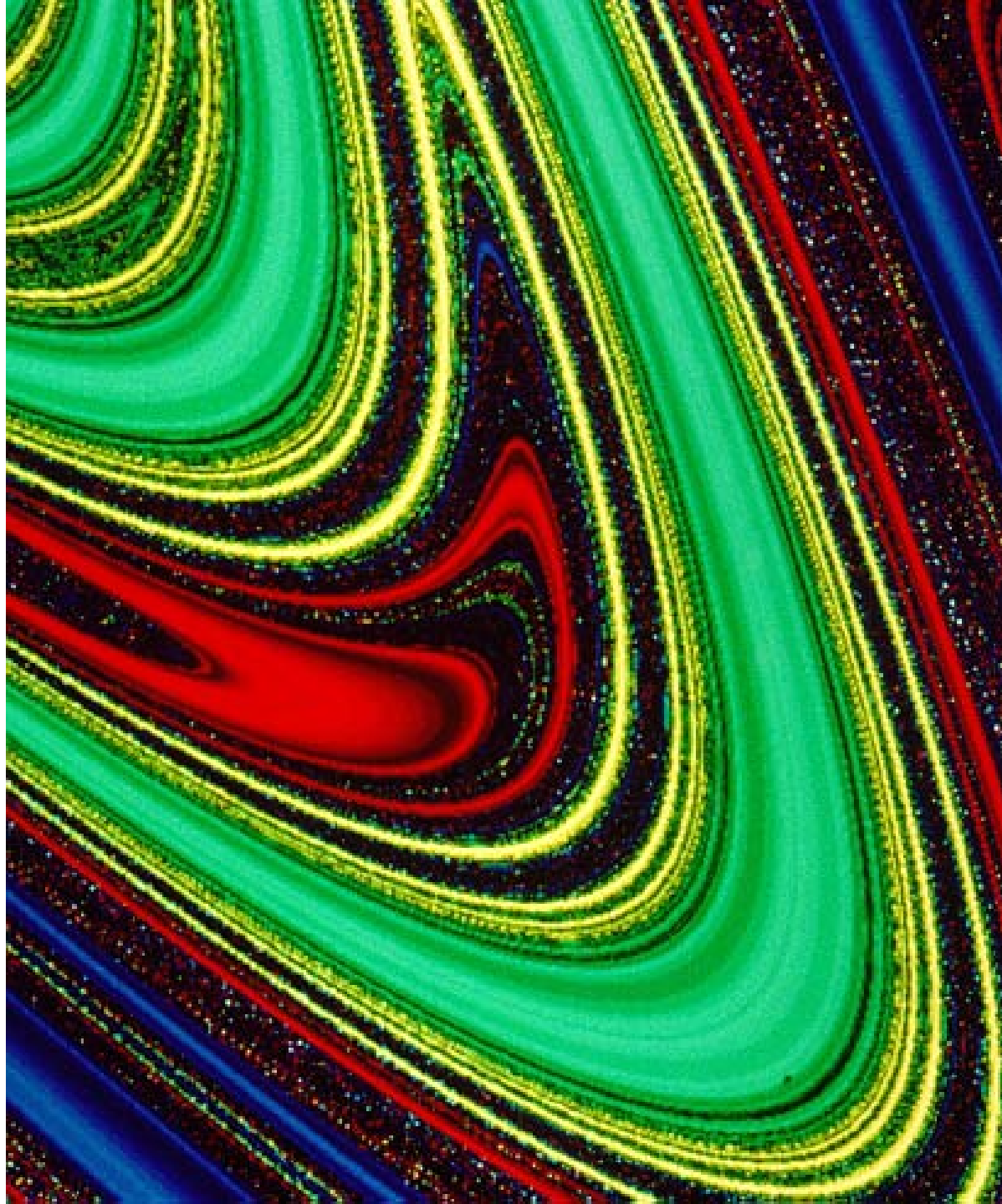
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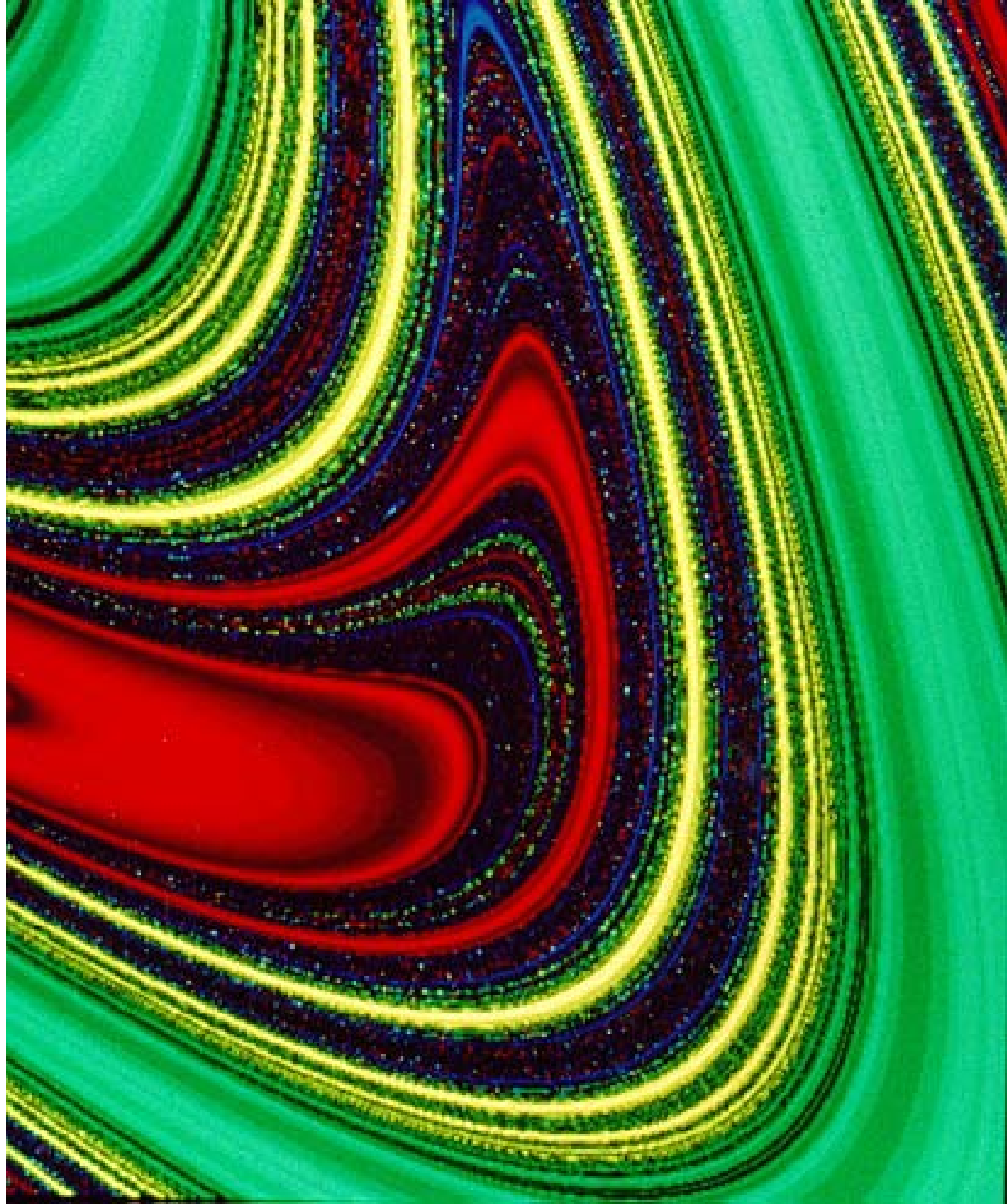


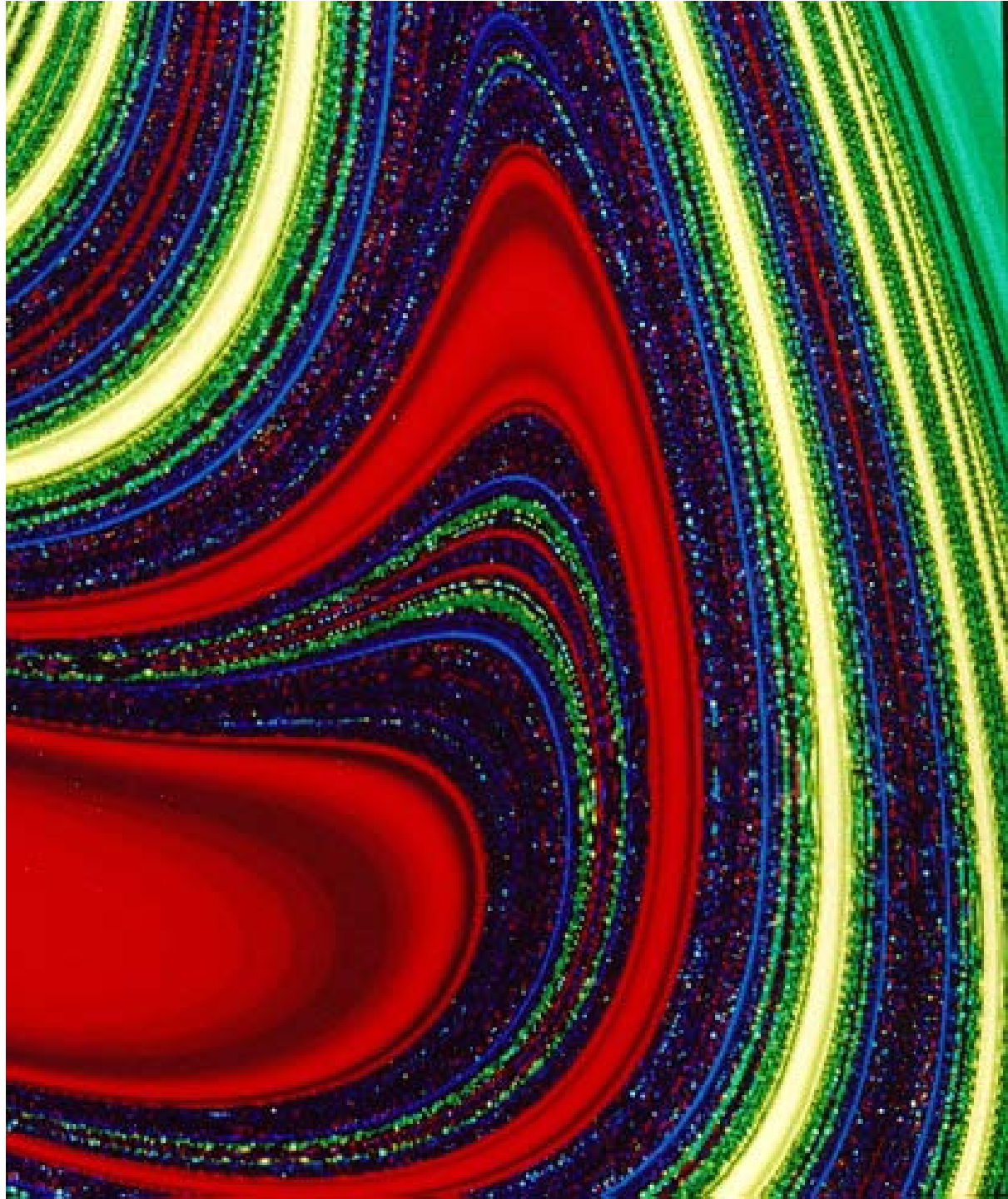


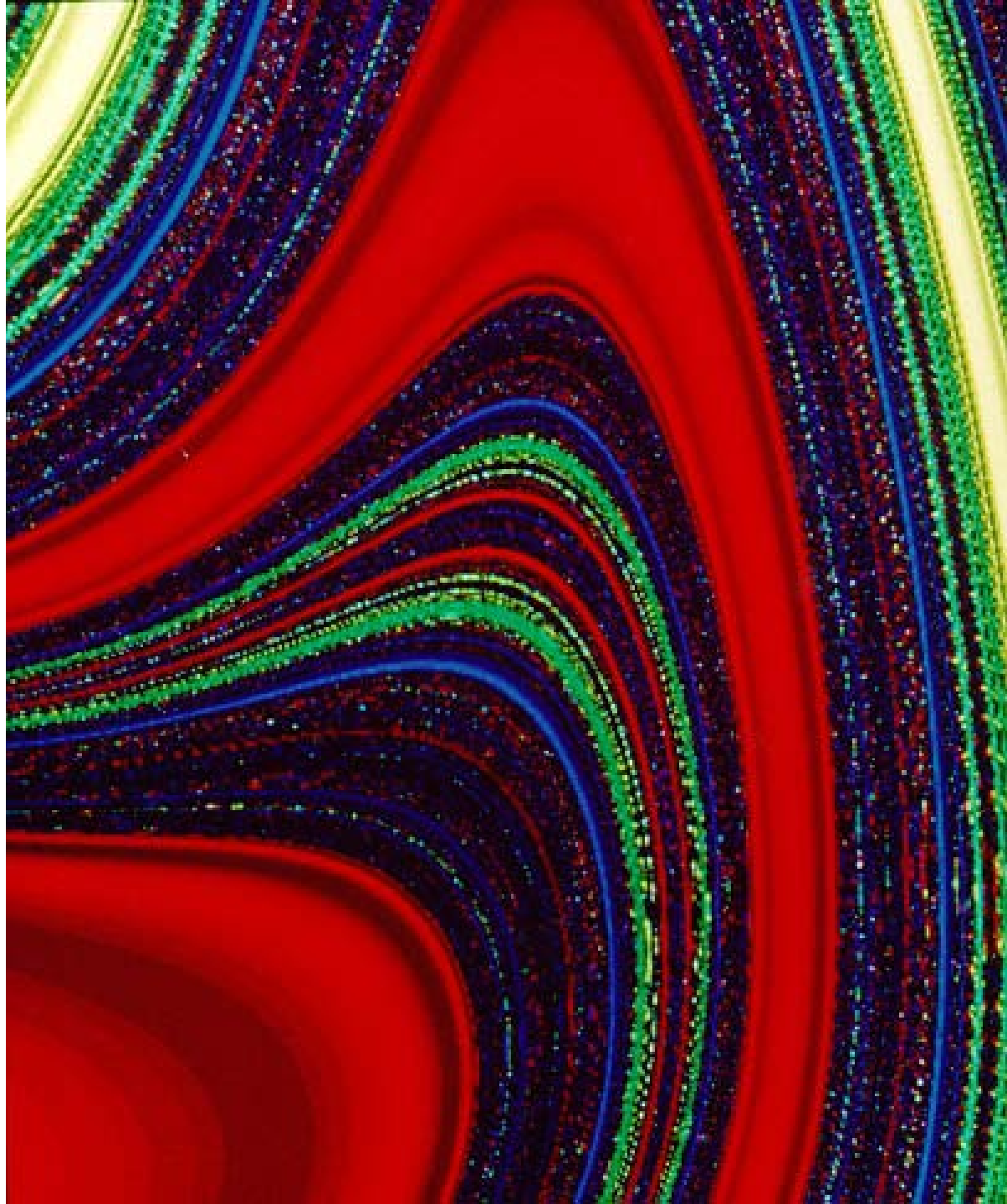


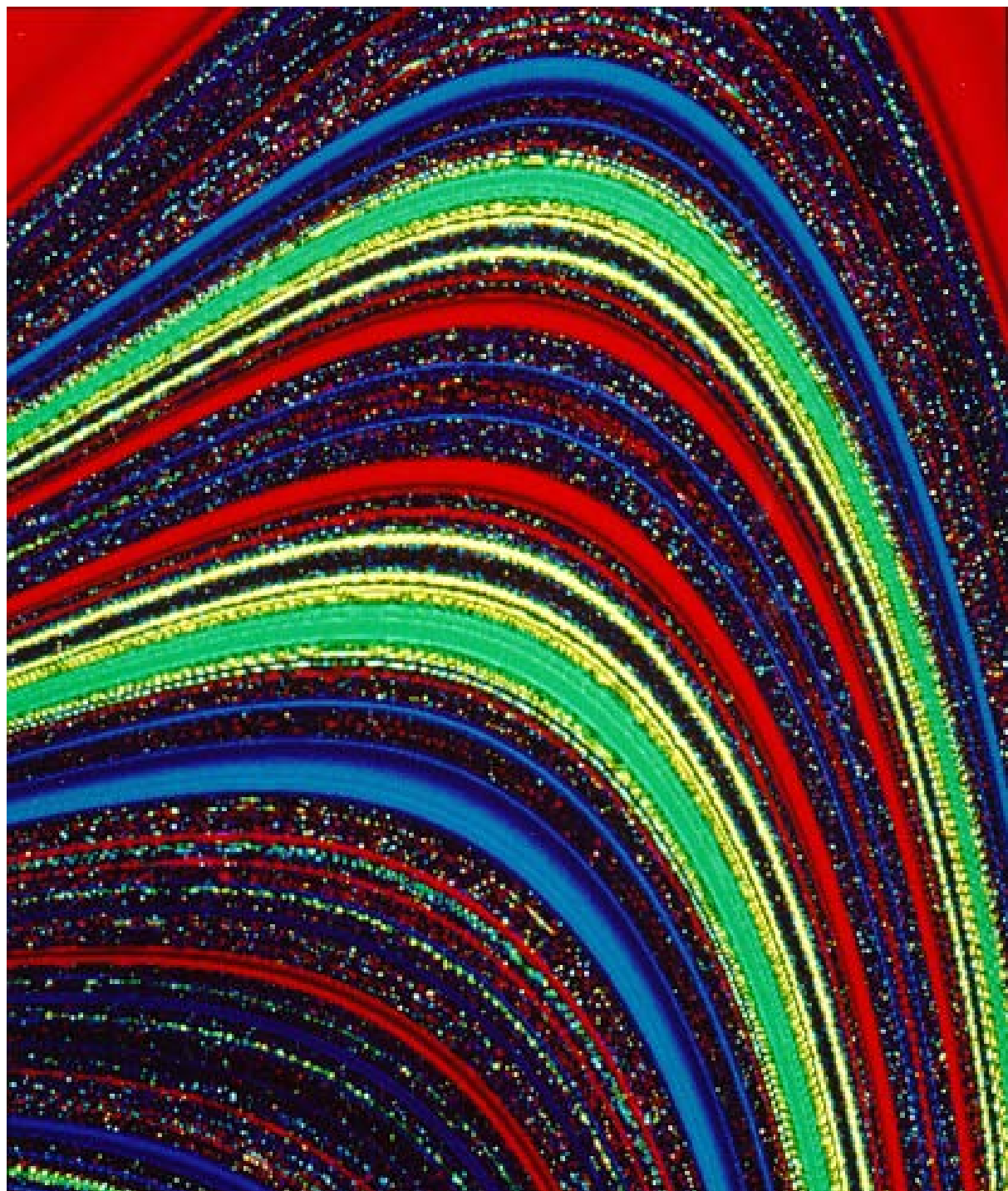


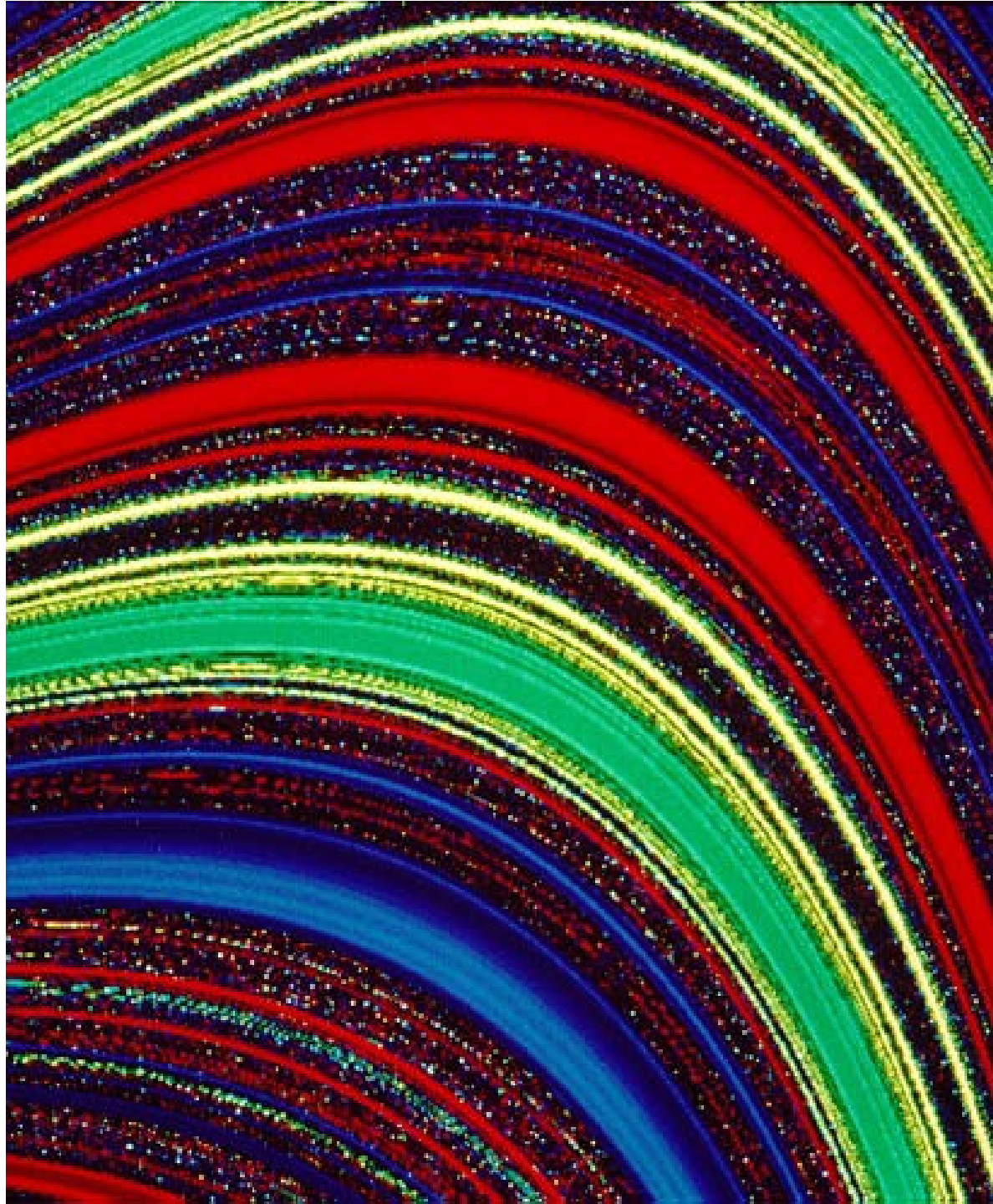


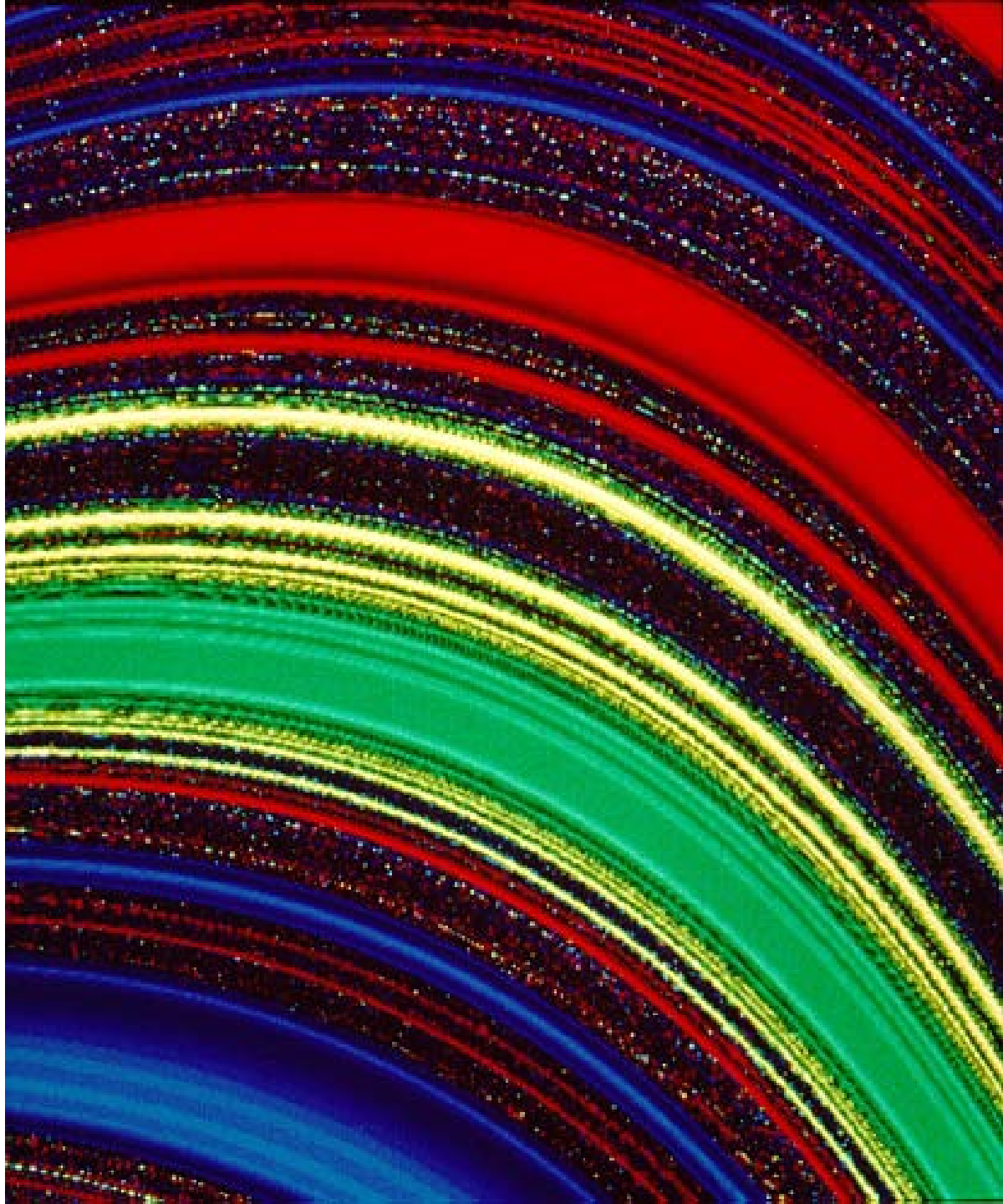


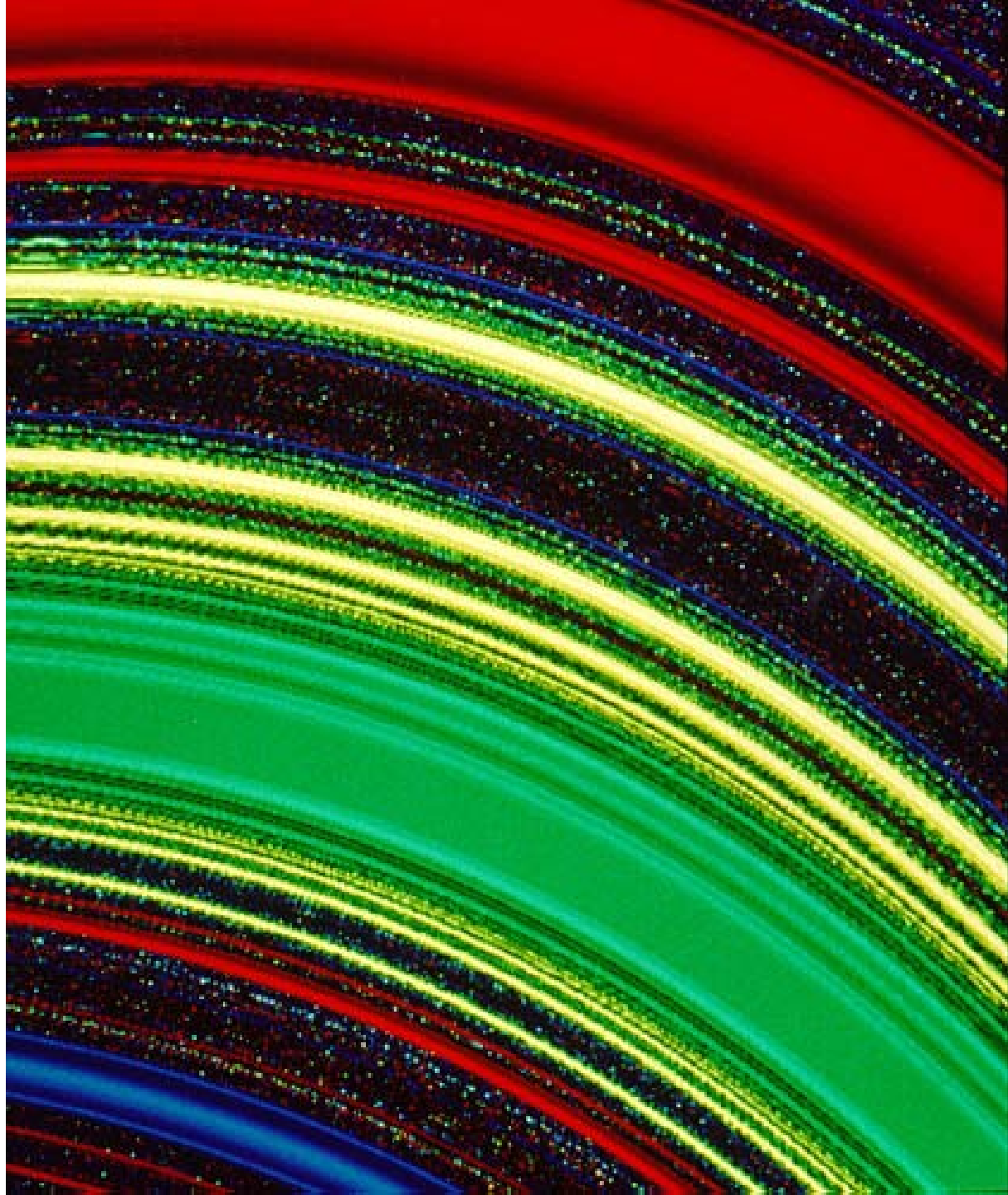


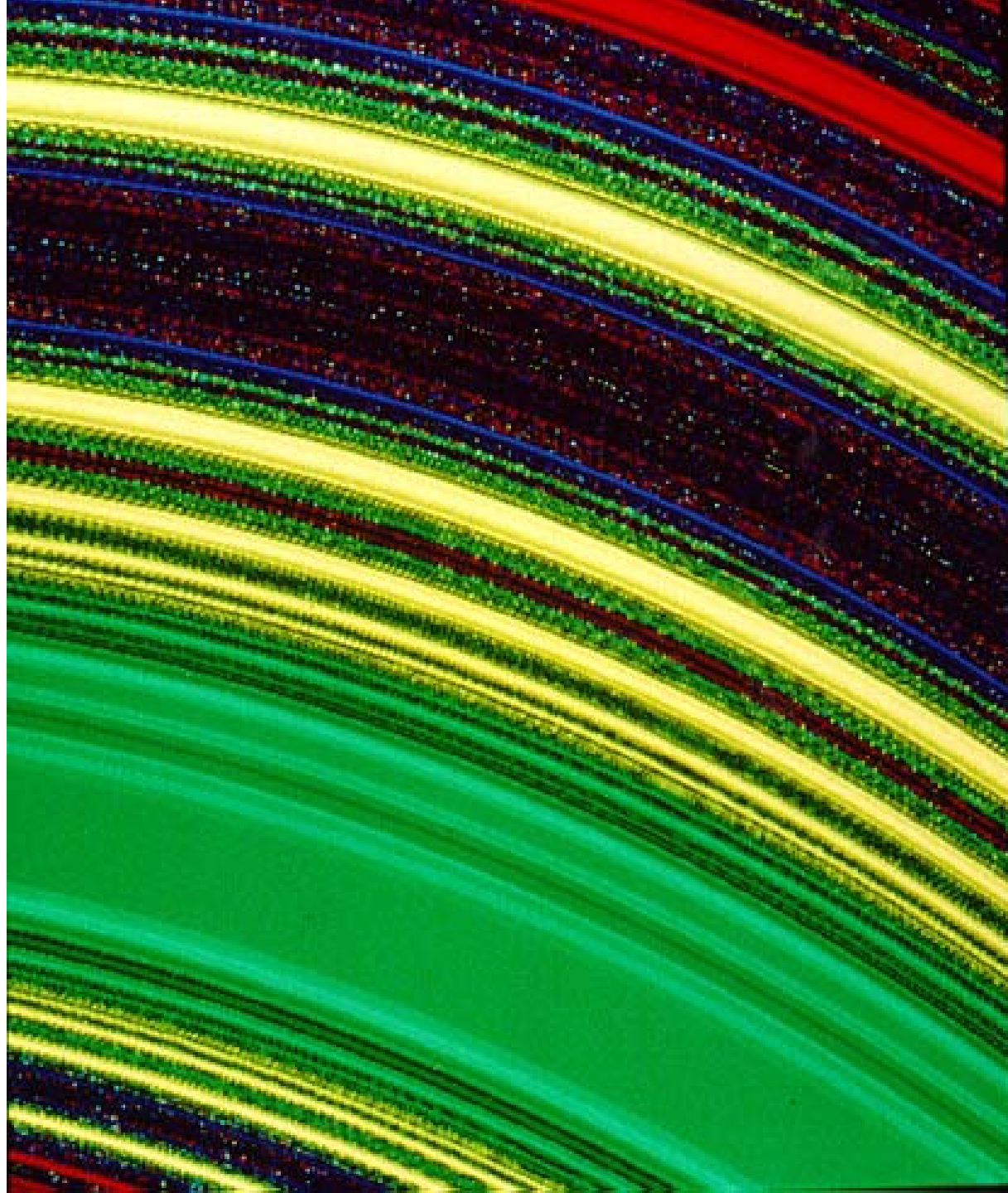


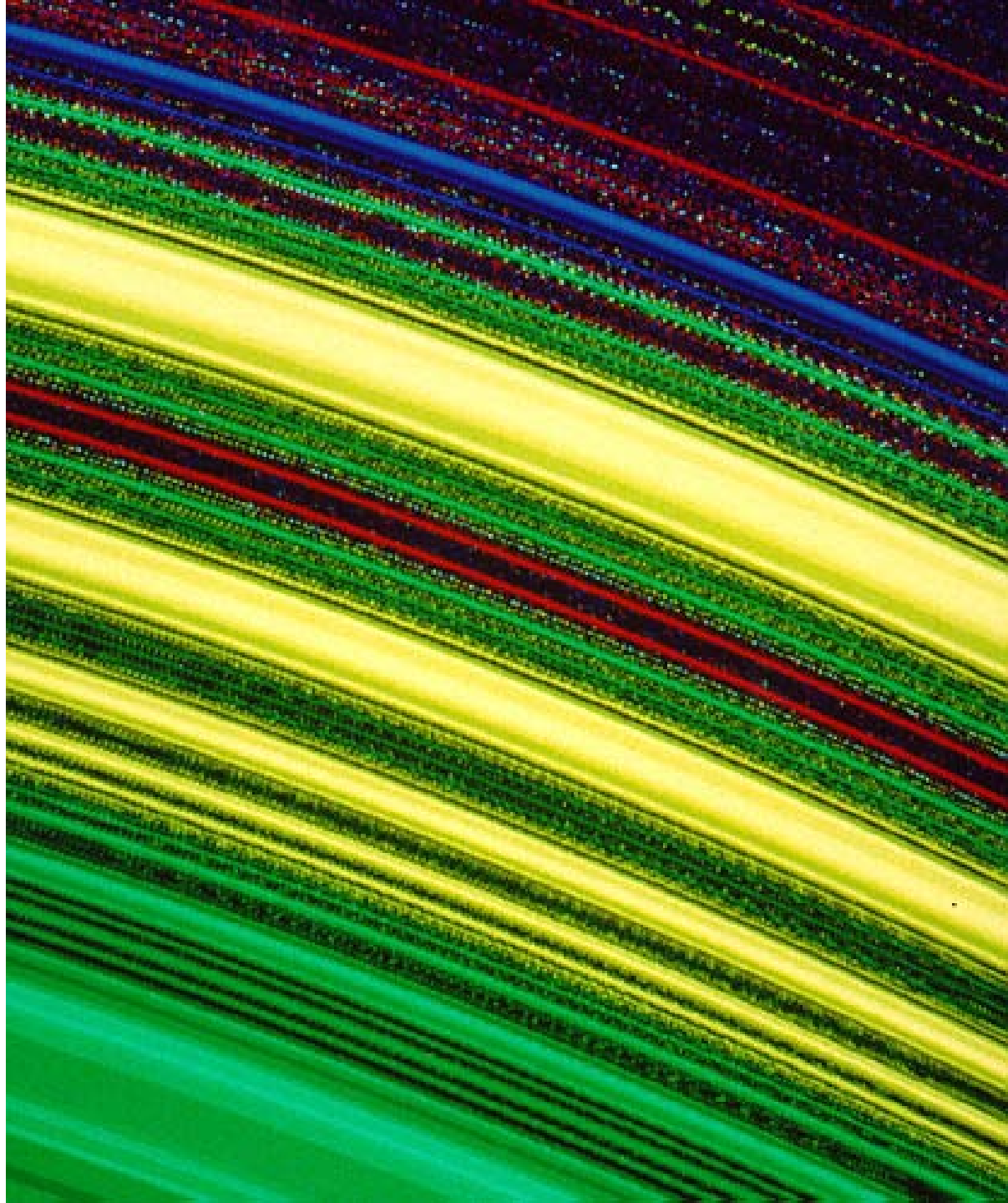


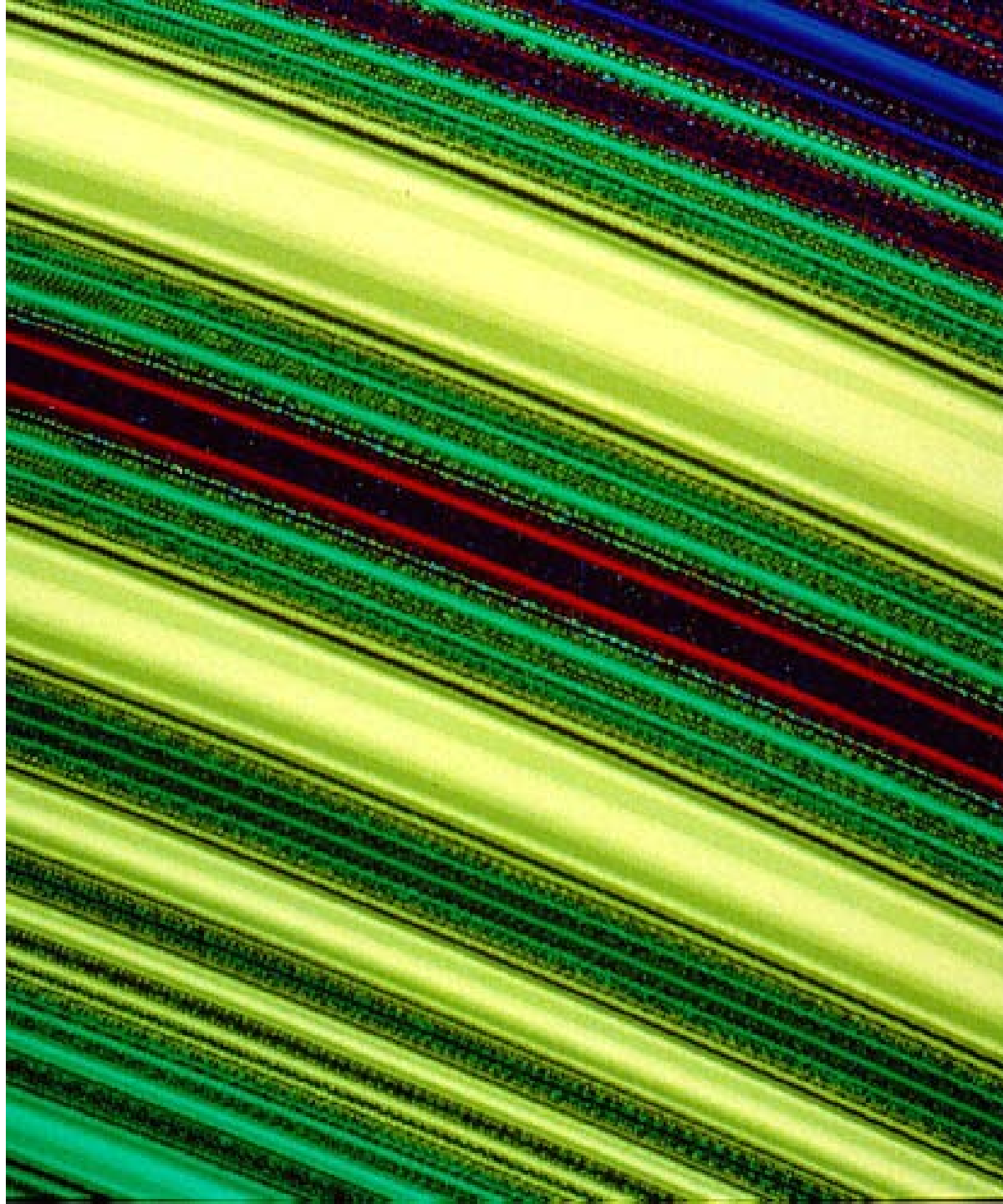




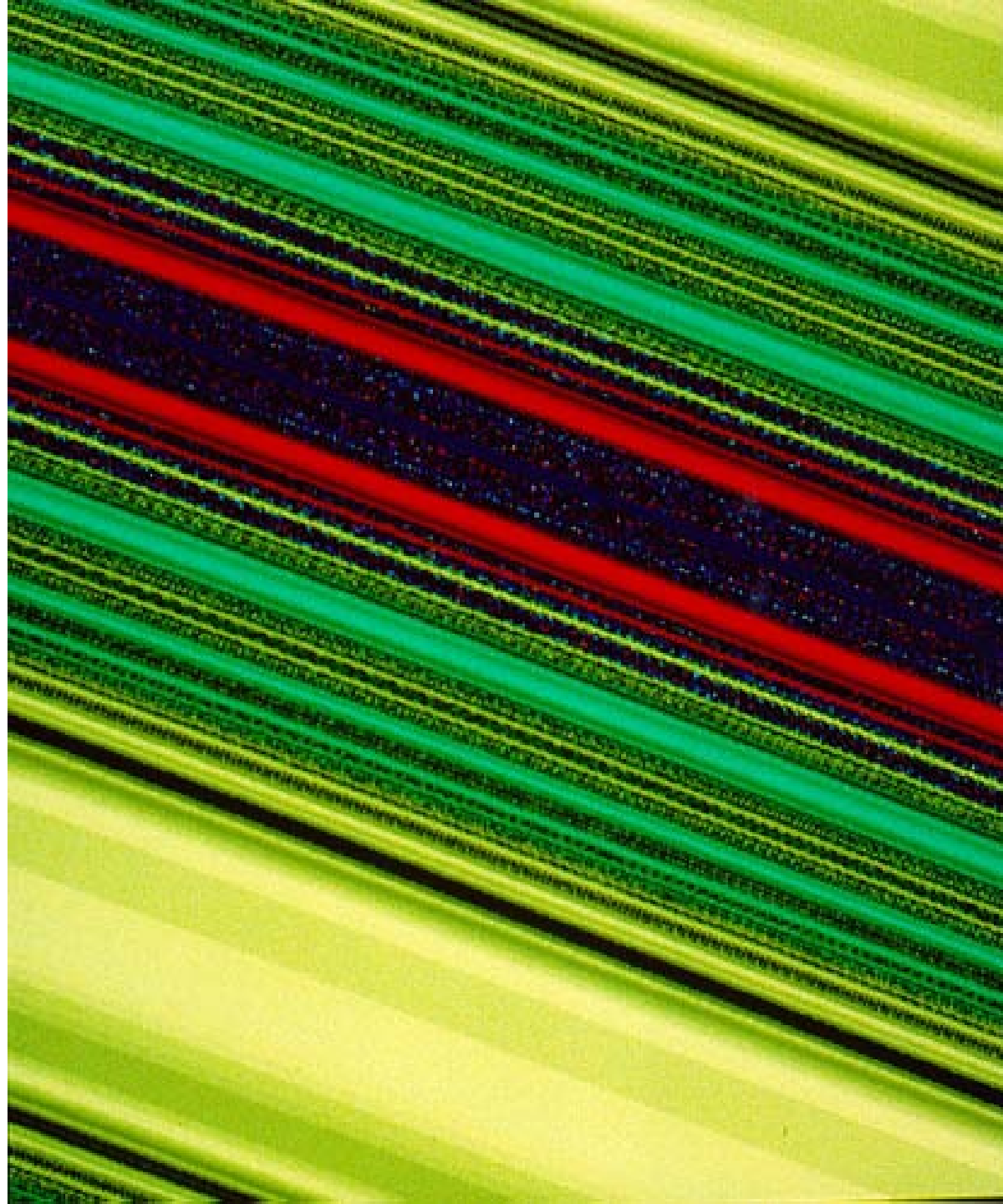


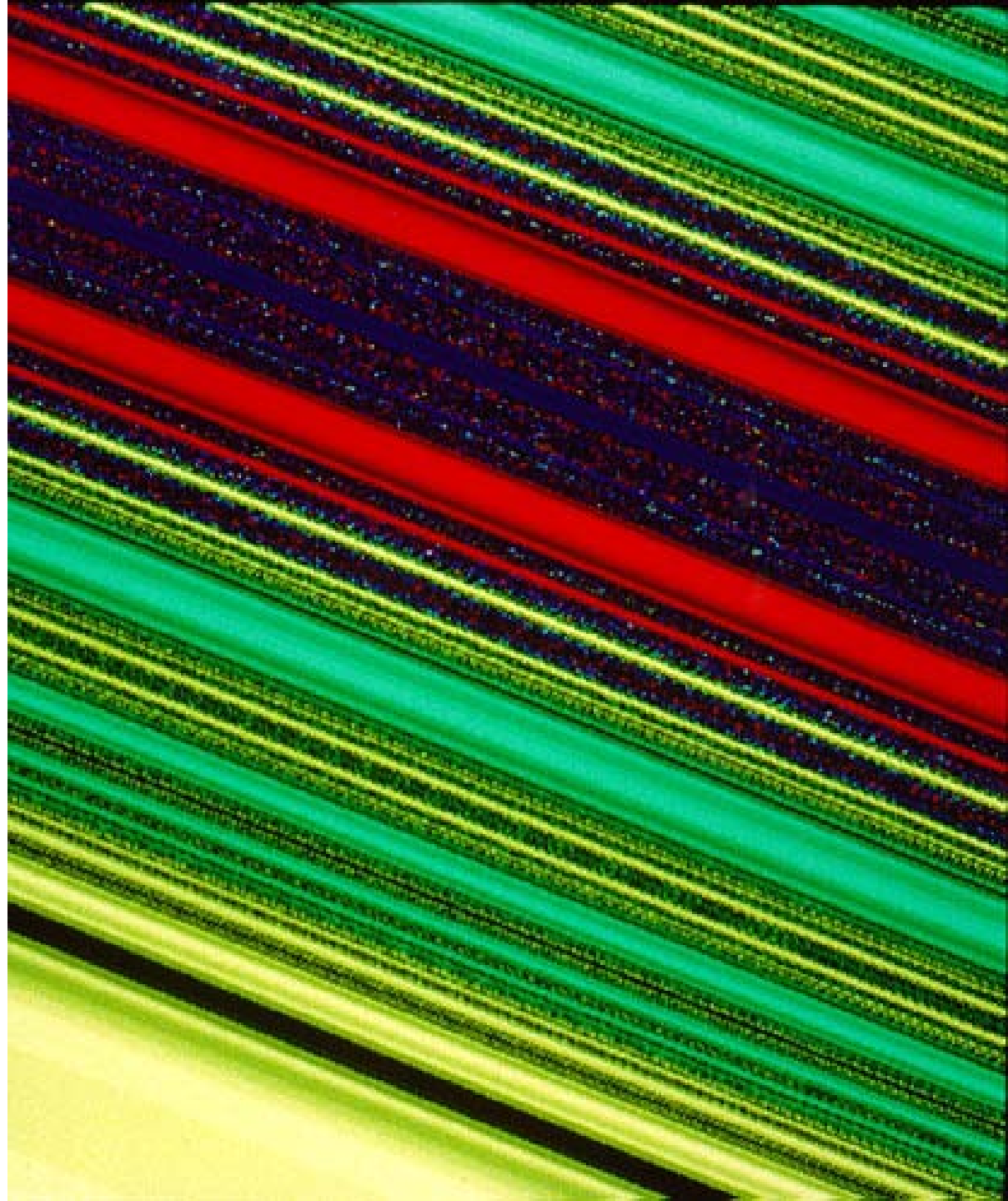


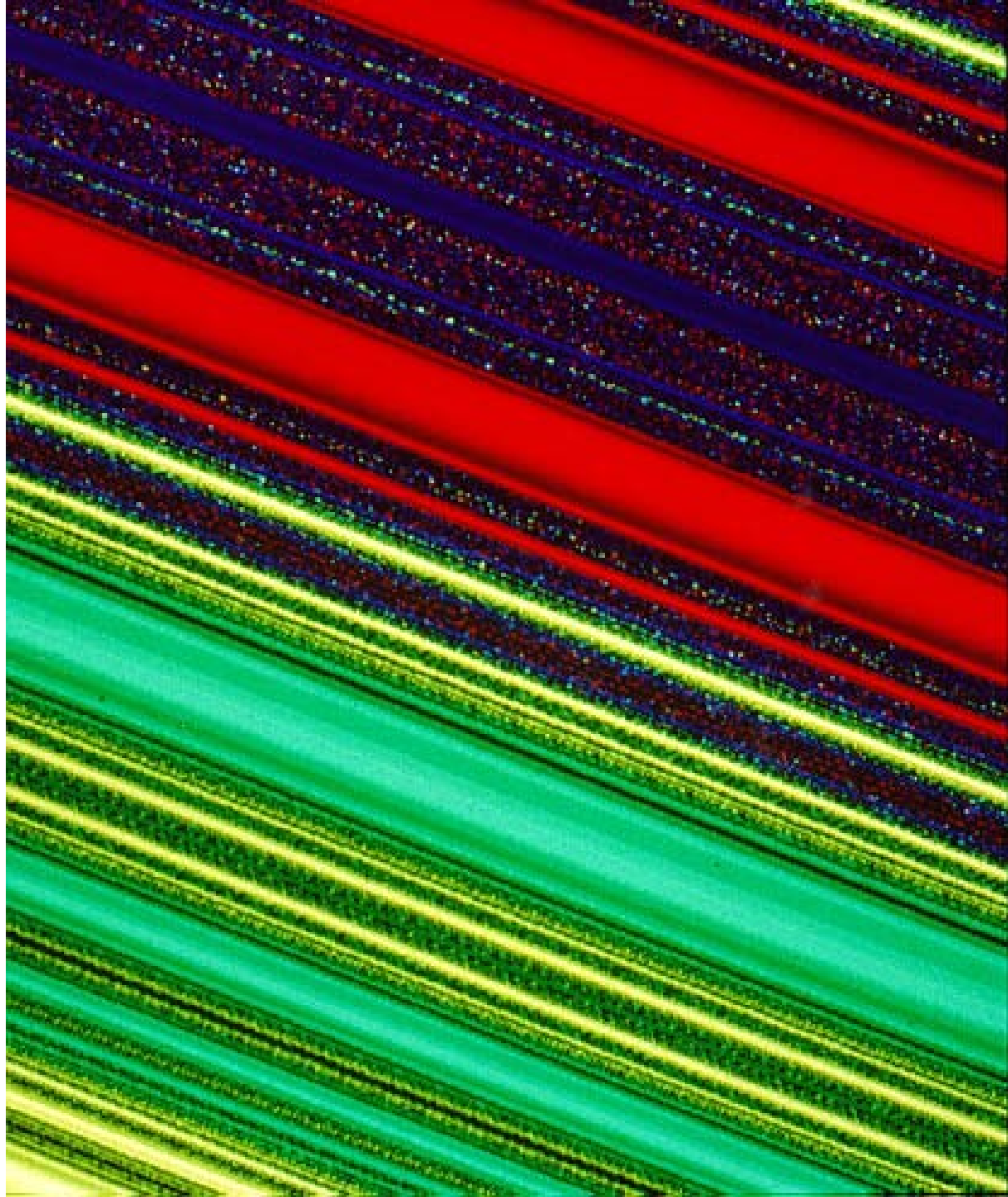


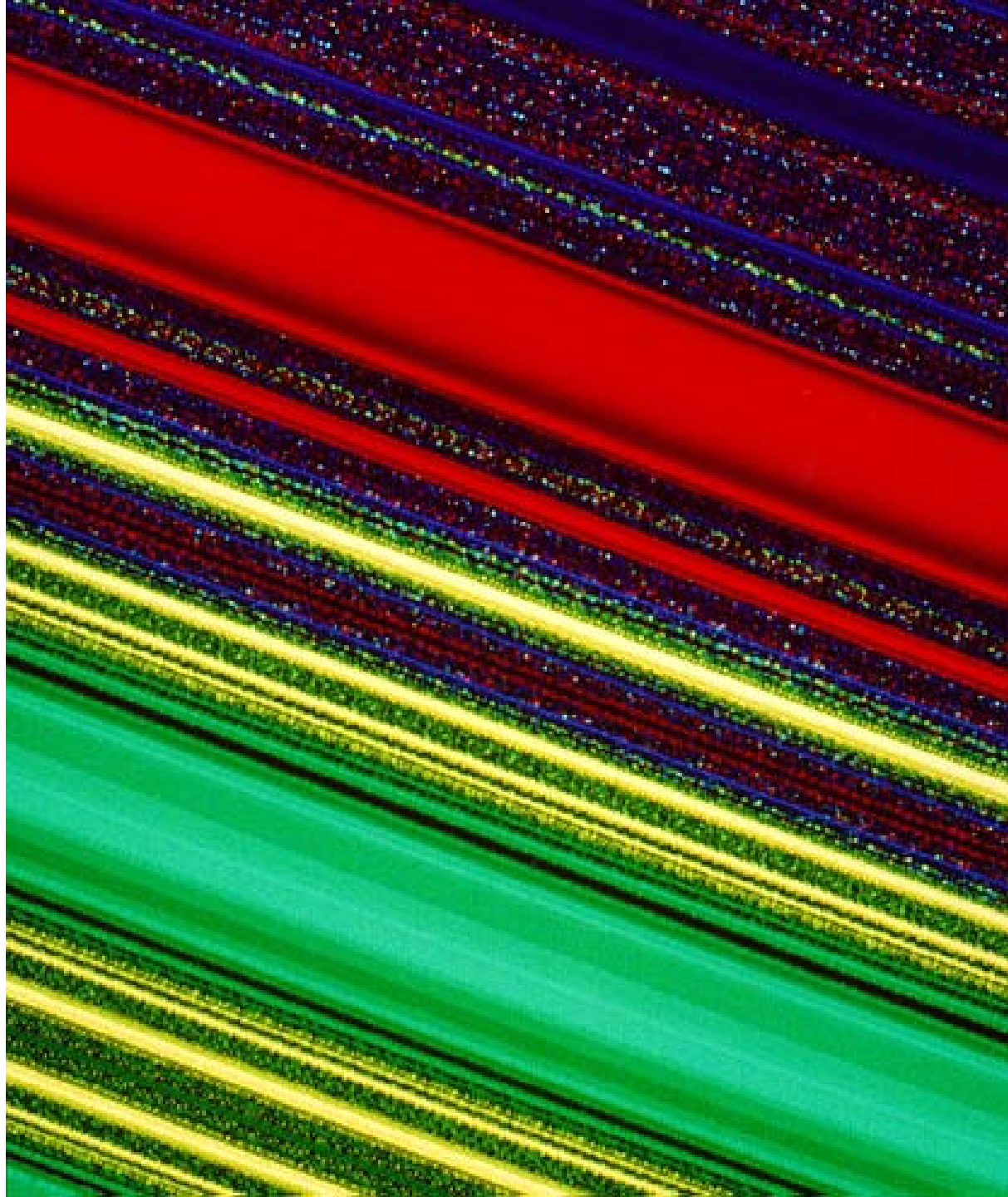


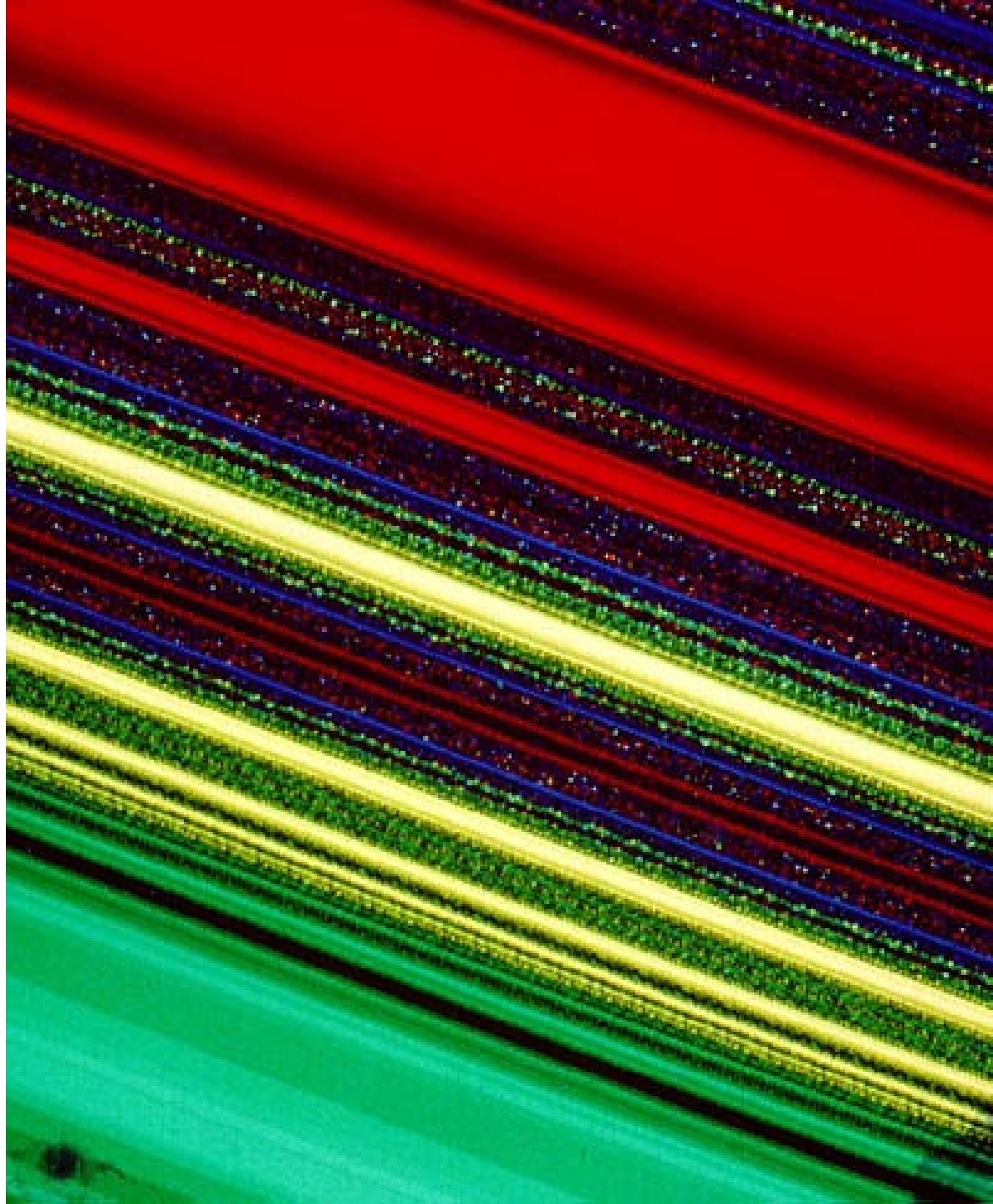






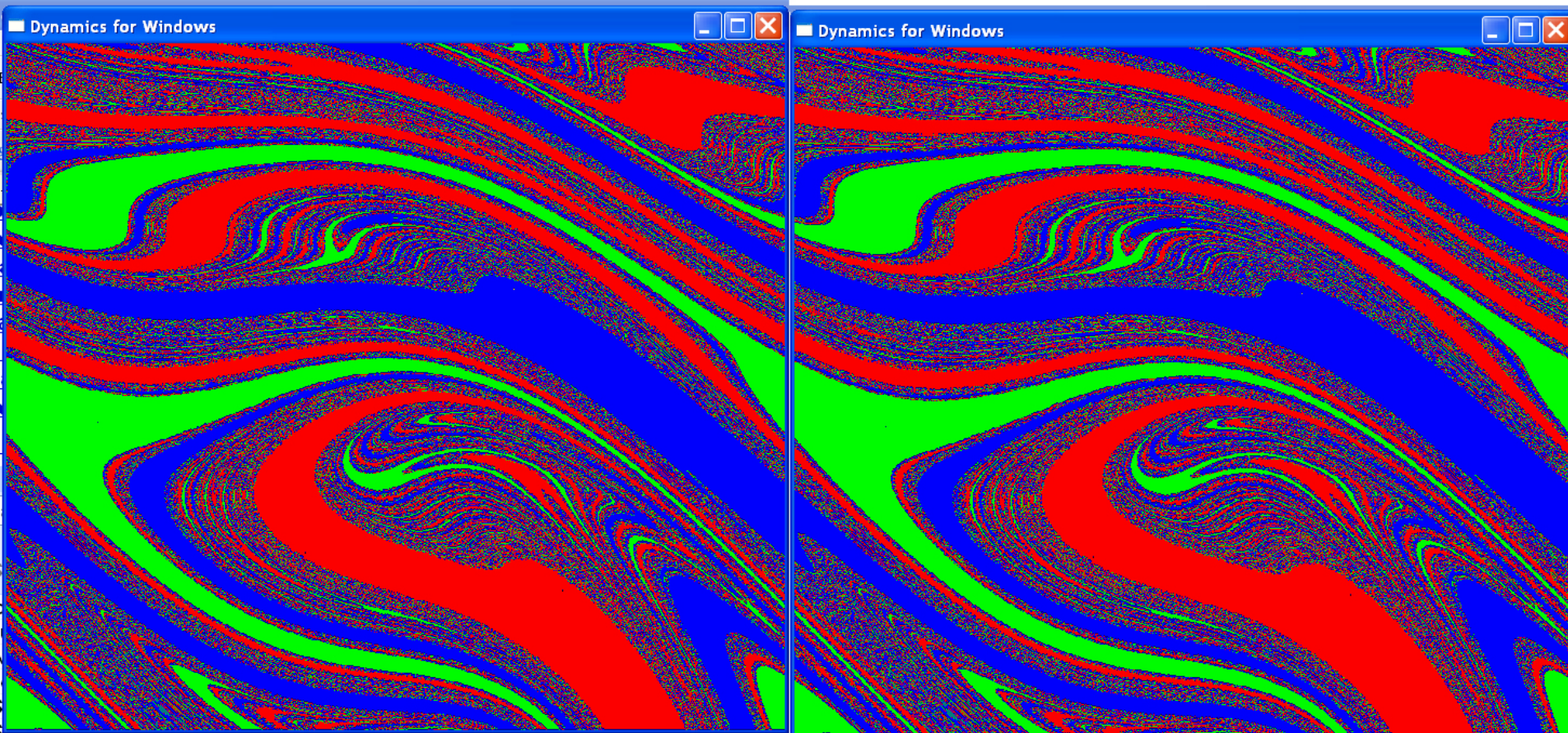




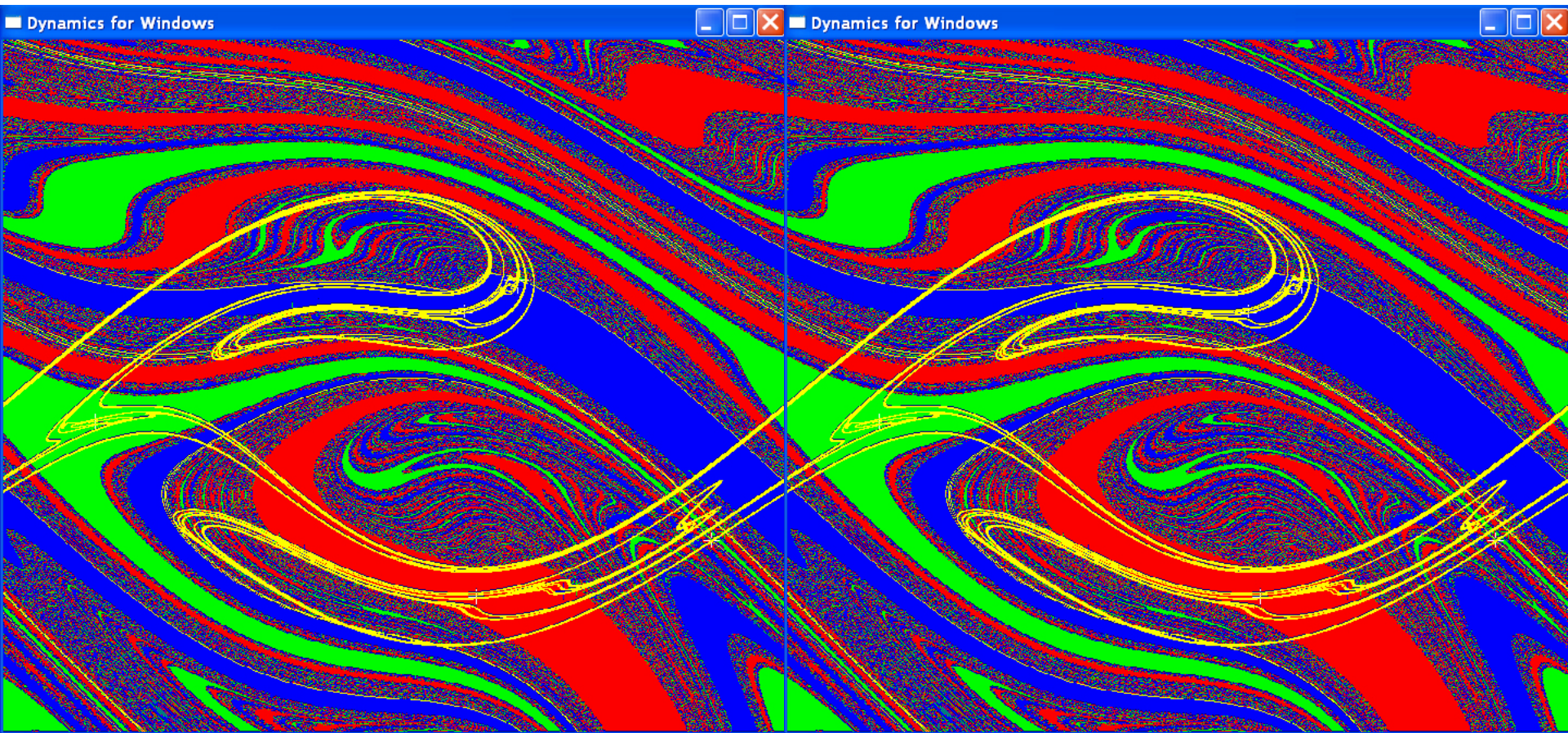


Fini
pendulum

Forced damped pendulum basins



Unstable manifold



Are the Solar System's major planets chaotic???

- The system: Sun + Jupiter + Saturn + Uranus + Neptune
- A report on the work of Wayne Hayes
- You can find the positions of these planets on the web. Neptune for example has an uncertainty in its position of 1000 km, far smaller than its diameter.
- Note: the diameter of Neptune's orbit is about 10^{16} mm.

Survey of chaos while varying Uranus semi-major axis, Quad Precision

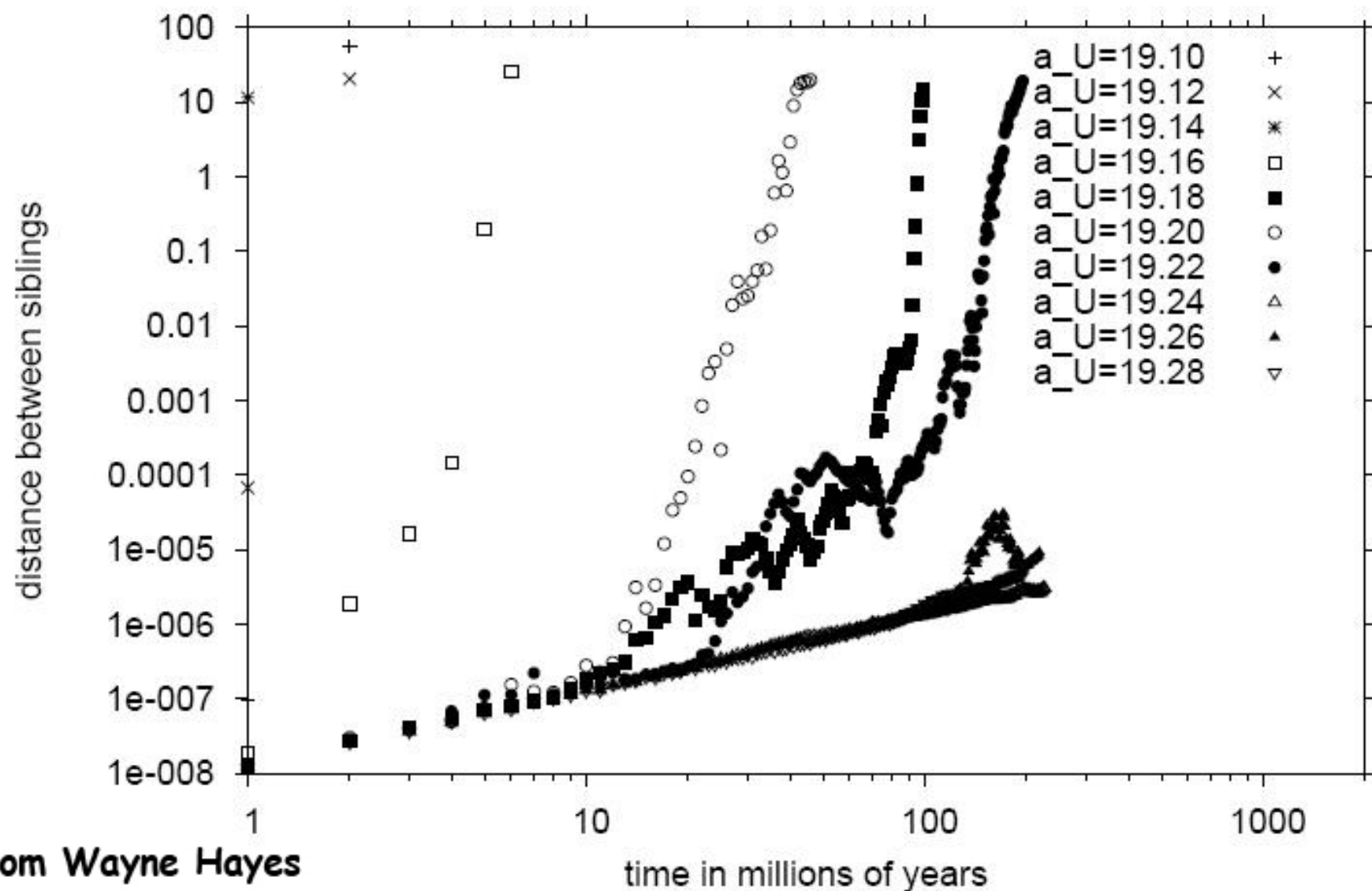


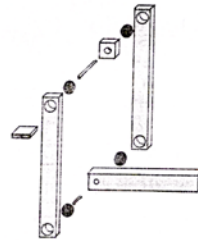
Figure 2: Murray + Holman (1999) in quadruple precision.

It is not easy; do
not make the
rotating parts tight.

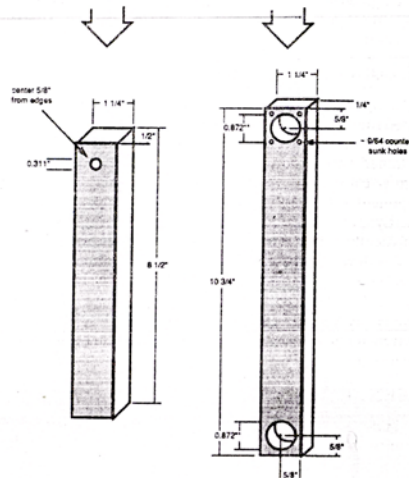
Parts list for one pendulum

Quantity	Item (all units are inches)
2	$\frac{1}{4} \times 1\frac{1}{4} \times 11$ aluminum bar stock
1	$\frac{1}{4} \times 1\frac{1}{4} \times 8\frac{1}{2}$ aluminum bar stock
1	$1 \times 1\frac{1}{4} \times 1\frac{1}{4}$ aluminum bar stock
1	$\frac{7}{16}$ o.d. $\times 1\frac{1}{2}$ CRS or stainless rod stock
2	$\frac{7}{16}$ o.d. $\times 3\frac{1}{2}$ CRS or stainless rod stock
1	$\frac{1}{4}$ i.d. E-rings
1	$1 \times 2 \times 2$ CRS bar stock
4	High precision bearings (e.g., roller skate bearings)
1	$\frac{3}{4}$ i.d. $\times \sim 3$ fiberglass or similar for sleeves (excess length allowed to fit in lathe chuck)
8	$\frac{1}{4} \times 8 - 32$ flat head machine screws

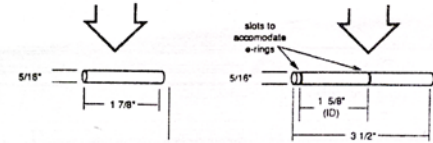
Assembly:
(sleeves, E-rings, screws not shown)



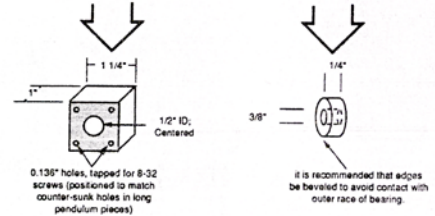
Material: Aluminum bar stock
Quantity: 1 Quantity: 2



Material: Cold Rolled Steel	Material: Cold Rolled Steel
Quantity: 1	Quantity: 1



Material: Aluminum	Material: Fiberglass or similar
Quantity: 1	Quantity: 2



Material: Cold Rolled Steel
Quantity: 1

